Exact Expressions for the Condition Number Distribution of Complex Wishart Matrices

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***Abstract*—The standard condition number (SCN) is a fundamental metric in the context of multiple-input multiple-output communication systems, linear detection and classical linear algebra. Hence, in this paper we propose a novel generic framework for the SCN distribution of three different classes of Wishart matrices which leads to new results and insights. In particular, our analysis covers both central and non-central Wishart distributions of arbitrary dimension and therefore is applicable to uncorrelated/semi-correlated Rayleigh fading and Ricean fading scenarios. For the special case of dual semi-correlated central Wishart matrices, we derive exact and asymptotic polynomial expressions for the SCN distributions. All analytical results are validated via Monte-Carlo simulations with the attained accuracy being excellent in all cases. The impact of the model parameters on channel conditioning is also investigated in detail.**

I. Introduction

The rapid development of multiple-input multiple-output (MIMO) systems over the past decade has triggered an extensive amount of research interest in random matrix theory (RMT). In fact, it has been demonstrated that the performance of practical MIMO transmission schemes are normally determined by the statistical eigenproperties of the instantaneous channel correlation matrix[1] $\mathbf{W} = \mathbf{H}\mathbf{H}^{\dagger}$ or $\mathbf{W} = \mathbf{H}^{\dagger}\mathbf{H}$, with $\mathbf{H}$ denoting the matrix of random MIMO channel gains. The matrix $\mathbf{H}$ is often modeled as a complex Gaussian matrix with mean and covariance structure defined by the system configuration (eg. antenna spacing) and the nature of the surrounding environment (eg. line-of-sight (LoS)). As such, $\mathbf{W}$ is said to follow a complex *Wishart* distribution.

Motivated by MIMO applications, many statistical properties of Wishart matrices have been studied in recent years. For instance, the unordered eigenvalue distributions and determinant properties of Wishart matrices have been addressed in [1]–[6] and applied to explore the ergodic capacity of the MIMO channel under different conditions. Ordered eigenvalue distributions have been derived in [7]–[13] and used to investigate the performance of MIMO beamforming strategies. For an excellent review of RMT and its application to wireless communications, see [14].

In addition, a critical metric in the area of RMT is the standard condition number (SCN), or the ratio of the largest to the smallest eigenvalue, which gives a measure of the relative conditioning (or rank-deficiency) of a matrix and has found numerous applications in a variety of mathematical fields (eg. sensitivity of matrix inversion schemes, convergence rate of iterative algorithms). In the MIMO context, the SCN has been used to indicate the degree of multipath activity in MIMO communication channels [15], [16], and has been shown to have a close connection with the performance of linear MIMO receivers in spatial multiplexing systems [17], [18].

Despite its importance and widespread use, significantly fewer results have been reported on the statistical properties of the SCN. This issue was first considered in the pioneering work of Edelman [19] whose SCN analysis focused on 2×2 (i.e. *dual*) uncorrelated central Wishart matrices with arbitrary degrees of freedom (DoF), and also on asymptotic distributions valid for infinitely large dimensions. In [20]–[22], the authors proposed simple efficient bounds on the high-tails of the SCN distributions, which again focused primarily on uncorrelated central Wishart matrices. The SCN probability density function (PDF) for the uncorrelated central case was recently presented in tensor form in [23], using the results of [24] for the ordered eigenvalues of Wishart matrices. The SCN analysis of [19] was extended in [25] to allow for matrices of arbitrary size and arbitrary correlation structures. However, the final result for the probability density function (PDF) of the SCN was given as a complicated expression involving an infinite series of zonal polynomials, and is thus difficult to manage. The exact SCN distributions for dual complex non-central Wishart matrices with two DoF were presented very recently in [26].

In this paper, we present a novel generic framework for the exact cumulative density function (CDF) of the SCN for different classes of complex Wishart matrices. This key result is a unified expression which applies for both central and non-central Wishart matrices of arbitrary dimension (for the central case, allowing for arbitrary correlation structure), and involves only a single scalar integral which allows for fast and efficient computation. For the case of dual Wishart matrices, we also derive new exact and asymptotic polynomial expressions for

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[1]The notation $(\cdot)^{\dagger}$ denotes the conjugate-transpose operation.

the SCN distribution. Based on these analytical results, we gain valuable insights into the statistical behavior of the channel conditioning for various MIMO fading scenarios, such as uncorrelated/semi-correlated Rayleigh fading and Ricean fading.

The remainder of the paper is organized as follows: In Section II, the MIMO channel models used throughout the paper are outlined while Section III covers some fundamental properties of the joint ordered eigenvalue distribution. In Section IV, the derivation of the generic SCN CDF framework is discussed. Section V elaborates on the specific case of dual Wishart matrices. In Section VI, the theoretical results are verified via Monte-Carlo simulations. Finally, Section VII concludes the paper and summarizes the key findings.

Notation: We use upper and lower case boldface to denote matrices and vectors, respectively. The $n \times n$ identity matrix is expressed as $\mathbf{I}_n$. The symbol $(\cdot)^T$ represents the transpose operation, $\mathrm{tr}(\cdot)$ yields the matrix trace, $\mathrm{etr}(\cdot)$ is shorthand for $\exp(\mathrm{tr}(\cdot))$, and $|\cdot|$ represents the determinant. In some cases, we will write the determinant of a matrix $\mathbf{A}$ in terms of its (i,j)-th elements (eg. as $|a_{i,j}|$). The symbol $\overset{\mathrm{d}}{\sim}$ denotes "distributed as", while $\sim$ will be used for asymptotic notation.

II. MIMO Channel Models

Let us consider a MIMO system equipped with N_t transmit and N_r receive antennas and also define $s \triangleq \min(N_t, N_r)$ and $t \triangleq \max(N_t, N_r)$. The wireless channel can be effectively characterized by the matrix $\mathbf{H} \in \mathbb{C}^{N_r \times N_t}$, whose entries represent the complex responses between each antenna pair. For most performance measures of interest, such as the ergodic and outage capacities [1]–[3], and the symbol error rate of practical transmission schemes [8]–[10], the effect of the channel is reflected via the statistical properties of the instantaneous MIMO correlation matrix

$$\mathbf{W} \triangleq \begin{cases} \mathbf{H}\mathbf{H}^\dagger, & \text{if } N_r \le N_t \\ \mathbf{H}^\dagger\mathbf{H}, & \text{if } N_r > N_t. \end{cases} \tag{1}$$

The channel matrix $\mathbf{H}$ is typically modeled as complex Gaussian. In this paper, we consider three common classes of channels, for which $\mathbf{W} \in \mathbb{C}^{s\times s}$ follows a Wishart distribution with t DoF. Note that, in all cases, the channel is normalized such that $E\left[\mathrm{tr}\left(\mathbf{W}\right)\right] = N_r N_t$. In the following, we will use $\mathbf{H}_w$ to represent a matrix whose entries are independent and identically distributed (i.i.d.) $\overset{\mathrm{d}}{\sim} \mathcal{CN}(0,1)$ random variables.

Definition 1 (Uncorrelated Rayleigh Fading): This model is valid when the antenna spacings and/or the angular spreads are high enough to induce independent fading, and there is no LoS path between the transmitter and receiver. Under these conditions,

$$\mathbf{H} = \mathbf{H}_w \tag{2}$$

and $\mathbf{W}$ is uncorrelated central Wishart with t DoF, commonly denoted as $\mathbf{W} \overset{\mathrm{d}}{\sim} \mathcal{CW}_s(t, \mathbf{I}_s)$ [27].

Definition 2 (Semi-Correlated Rayleigh Fading): In practice, the MIMO spatial subchannels are often correlated due to the limited angular spreads or restrictions on the array sizes. The effect of spatial correlation can be reflected as follows

$$\mathbf{H} = \begin{cases} \mathbf{\Sigma}_s^{1/2}\mathbf{H}_w, & \text{if } N_r \le N_t \\ \mathbf{H}_w\mathbf{\Sigma}_s^{1/2}, & \text{if } N_r > N_t \end{cases} \tag{3}$$

where $\mathbf{\Sigma}_s \in \mathbb{C}^{s\times s} > \mathbf{0}$ denotes the spatial correlation matrix. Note that here we have assumed that correlation occurs only at the side with the minimum number of antennas[2], in which case $\mathbf{W}$ is semi-correlated central Wishart, expressed as $\mathbf{W} \overset{\mathrm{d}}{\sim} \mathcal{CW}_s(t, \mathbf{\Sigma}_s)$.

Definition 3 (Uncorrelated Ricean Fading): In the case of Ricean fading, the wireless link between the two ends is dominated by a LoS component. Hence, the channel matrix $\mathbf{H} \in \mathbb{C}^{N_r\times N_t}$ consists of a deterministic component, $\mathbf{H}_{\mathrm{L}}$, and a random component, $\mathbf{H}_w$, which accounts for the scattered signals. The channel model is then written as

$$\mathbf{H} = \sqrt{\frac{K_r}{K_r+1}}\mathbf{H}_{\mathrm{L}} + \sqrt{\frac{1}{K_r+1}}\mathbf{H}_w \tag{4}$$

where K_r stands for the Ricean K-factor, representing the ratio of the power of the deterministic component to the power of the fading component. Clearly, $\mathbf{H}$ has a non-zero mean, given by $E\left\{\mathbf{H}\right\} = \mathbf{M} = \sqrt{K_r/(K_r+1)}\mathbf{H}_{\mathrm{L}}$. Then, $\mathbf{W}$ is uncorrelated non-central Wishart, expressed as $\mathbf{W} \overset{\mathrm{d}}{\sim} \mathcal{CW}_s\left(t, \epsilon^2\mathbf{I}_s, \mathbf{\Omega}\right)$, where $\varepsilon = 1/\sqrt{K_r+1}$ is a power scaling factor, and

$$\mathbf{\Omega} = \begin{cases} \epsilon^{-2}\mathbf{M}\mathbf{M}^\dagger, & \text{if } N_r \le N_t \\ \epsilon^{-2}\mathbf{M}^\dagger\mathbf{M}, & \text{if } N_r > N_t \end{cases} \tag{5}$$

is the non-centrality matrix. For the sake of consistency with previous studies, we consider a scaled version of $\mathbf{W}$, that is $\mathbf{S} = \epsilon^{-2}\mathbf{W} \overset{\mathrm{d}}{\sim} \mathcal{CW}_s\left(t, \mathbf{I}_s, \mathbf{\Omega}\right)$ [8], [9], [27]. Note that $\mathbf{S}$ and $\mathbf{W}$ have, by definition, the same SCN.

III. Joint Eigenvalue PDF and SCN

The MIMO correlation matrix $\mathbf{W}$ (or $\mathbf{S}$) is Hermitian and positive definite, therefore its eigenvalues are real and positive. Let us denote these eigenvalues by the vector $\boldsymbol{\lambda} \triangleq [\lambda_1, \lambda_2, \ldots, \lambda_s]^T$, with $\lambda_1 \ge \lambda_2 \ldots \ge \lambda_s > 0$. For each of the three channel models introduced in Definitions 1–3, the corresponding joint ordered eigenvalue distributions admit the following generic form [27, Eq. (102)], [8, Eq. (45)], and more recently [5], [11], [12]

$$f(\boldsymbol{\lambda}) = K|\mathbf{\Phi}(\boldsymbol{\lambda})| \times |\mathbf{\Psi}(\boldsymbol{\lambda})| \prod_{\ell=1}^{s} \xi(\lambda_\ell). \tag{6}$$

Table I tabulates the parameters for each of the channel scenarios considered where $\mathbf{V}_1(\boldsymbol{\lambda})$ is a Vandermode matrix with (i,j)-th entry λ_j^{i-1}; the vectors $\boldsymbol{\sigma} \triangleq [\sigma_1, \sigma_2, \ldots, \sigma_s]^T$ and $\boldsymbol{\omega} \triangleq [\omega_1, \omega_2, \ldots, \omega_s]^T$ contain the non-zero ordered eigenvalues (assumed distinct) of the matrices $\mathbf{\Sigma}_s$ and $\mathbf{\Omega}$, respectively, i.e. $\sigma_1 > \sigma_2 \ldots > \sigma_s > 0$ and $\omega_1 > \omega_2 \ldots > \omega_s > 0$. Also, $\mathbf{F}(\boldsymbol{\lambda}, \boldsymbol{\omega})$ and $\mathbf{E}(\boldsymbol{\lambda}, \boldsymbol{\sigma})$ have (i,j)-th entries

[2]Correlation at the side with the maximum number of antennas can also be handled, however the notation is more cumbersome. Thus, we choose to omit the explicit presentation of this scenario throughout the paper.

TABLE I

JOINT EIGENVALUE PDF OF COMPLEX WISHART MATRICES: PARAMETERS AND NORMALIZATION CONSTANTS

	$\boldsymbol{\Phi}(\boldsymbol{\lambda})$	$\boldsymbol{\Psi}(\boldsymbol{\lambda})$	$\xi(\lambda_\ell)$	K
Uncorrelated central (UC)	$\mathbf{V}_1(\boldsymbol{\lambda})$	$\mathbf{V}_1(\boldsymbol{\lambda})$	$\lambda_\ell^{t-s}e^{-\lambda_\ell}$	$K_{uc}=\left[\prod_{i=1}^{s}(s-i)!\prod_{j=1}^{s}(t-j)!\right]^{-1}$
Semi-correlated central (CC)	$\mathbf{V}_1(\boldsymbol{\lambda})$	$\mathbf{E}(\boldsymbol{\lambda},\boldsymbol{\sigma})$	λ_ℓ^{t-s}	$K_{cc}=\prod_{i=1}^{s}\frac{1}{\sigma_i^t(t-i)!}\prod_{i<j}^{s}\frac{\sigma_i\sigma_j}{\sigma_j-\sigma_i}$
Uncorrelated non-central (UN)	$\mathbf{V}_1(\boldsymbol{\lambda})$	$\mathbf{F}(\boldsymbol{\lambda},\boldsymbol{\omega})$	$\lambda_\ell^{t-s}e^{-\lambda_\ell}$	$K_{un}=(t-s)!^{-s}\mathrm{etr}(-\boldsymbol{\omega})\lvert\mathbf{V}_1(\boldsymbol{\omega})\rvert^{-1}$

${}_0\mathcal{F}_1(t-s+1;\lambda_j\omega_i)$ and $\exp(-\lambda_j/\sigma_i)$ respectively, with ${}_0\mathcal{F}_1(\cdot;\cdot)$ the standard generalized hypergeometric function [28, Eq. (9.14.1)]. In all cases, the (i,j)-th element of $\boldsymbol{\Phi}(\boldsymbol{\lambda})$ and $\boldsymbol{\Psi}(\boldsymbol{\lambda})$ has the same general form:

$$\{\boldsymbol{\Phi}(\boldsymbol{\lambda})\}_{i,j}=\phi_i(\lambda_j),\qquad \{\boldsymbol{\Psi}(\boldsymbol{\lambda})\}_{i,j}=\psi_i(\lambda_j) \tag{7}$$

with $\phi_i(\cdot)$ and $\psi_i(\cdot)$ defined according to the specific channel scenario, as described above

Apart from the marginal/joint eigenvalue distribution, a critical metric in the context of RMT is the SCN, or

$$z\triangleq\frac{\lambda_1}{\lambda_s},\qquad z\geq 1. \tag{8}$$

We recall that a condition number close to one indicates a well-conditioned full-rank matrix with almost equal eigenvalues, whereas a very high condition number implies a near rank-deficient matrix.

IV. GENERIC FRAMEWORK FOR THE SCN CDF

As was previously stated, the main scope of this paper is to provide novel closed-form expressions for the SCN CDF of three different classes of Wishart matrices. The following theorem provides a unified framework for the SCN CDF which applies for arbitrary matrix dimensions and arbitrary DoF.

Theorem 1: The CDF, $F_z(x)$, of the SCN of an uncorrelated central Wishart matrix, $\mathbf{W}\overset{\mathrm{d}}{\sim}\mathcal{CW}_s(t,\mathbf{I}_s)$, semi-correlated central Wishart matrix, $\mathbf{W}\overset{\mathrm{d}}{\sim}\mathcal{CW}_s(t,\boldsymbol{\Sigma}_s)$, and uncorrelated non-central Wishart matrix, $\mathbf{S}\overset{\mathrm{d}}{\sim}\mathcal{CW}_s(t,\mathbf{I}_s,\boldsymbol{\Omega})$, can be represented as

$$F_z(x)=K\sum_{\ell=1}^{s}\int_0^\infty\left|\left[\begin{array}{ll}\int_{\lambda_s}^{x\lambda_s}\phi_i(u)\psi_j(u)\xi(u)du, & i\neq\ell\\ \phi_i(\lambda_s)\psi_j(\lambda_s)\xi(\lambda_s), & i=\ell\end{array}\right]\right|d\lambda_s \tag{9}$$

with $\phi_i(\cdot)$ and $\psi_i(\cdot)$ defined as in (7), while K is the normalization constant given in Table I.

Proof: A sketch of the proof is given in Appendix A while a detailed derivation is provided in an extended journal version of this paper [29]. ■

It is worth mentioning that all integrals inside the determinant admit a closed-form solution, for all cases of interest. Thus, only a *single integration* is required, whose numerical evaluation is more robust and efficient compared to conventional Monte-Carlo simulations. The closed-form solutions for the definite integrals in (9) are summarized in Table II, where we have made use of [28, Eq. (3.381.1)] and [8, Eq. (4)]. In these expressions, $\gamma(a,x)$ denotes the lower incomplete gamma function [28, Eq. (8.350.1)] and $Q_{p,q}(a,b)$ is the Nuttall Q-function defined as [30]

TABLE II

CLOSED-FORM EXPRESSIONS FOR THE DEFINITE INTEGRAL IN (9)

	$\mathcal{I}_{i,j}=\int_{\lambda_s}^{x\lambda_s}\phi_i(u)\psi_j(u)\xi(u)du$
UC	$\gamma(t-s+i+j-1,x\lambda_s)-\gamma(t-s+i+j-1,\lambda_s)$
CC	$\left(\gamma(t-s+i,x\lambda_s/\sigma_j)-\gamma(t-s+i),\lambda_s/\sigma_j)\right)\sigma_j^{t-s+i}$
UN	$(t-s)!e^{\omega_j}\omega_j^{(s-t)/2}2^{(s-t-2i+2)/2}\times\Big(Q_{t-s+2i-1,t-s}\left(\sqrt{2\omega_j},\sqrt{2\lambda_s}\right)-Q_{t-s+2i-1,t-s}\left(\sqrt{2\omega_j},\sqrt{2x\lambda_s}\right)\Big)$

$$Q_{p,q}(a,b)=\int_b^\infty x^p\exp\left(-\frac{x^2+a^2}{2}\right)I_q(ax)dx \tag{10}$$

with $I_q(\cdot)$ denoting the q-th order modified Bessel function of the first kind. Note that since the sum of the indices of the Nuttall-Q function in Table II is always odd, a closed-form representation is given in [31, Eq. (13)] in terms of Marcum Q-functions and a finite weighted sum of modified Bessel functions of the first kind.

An alternative expression for the uncorrelated non-central case can also be obtained by expanding the involved hypergeometric function as follows [28, Eq. (9.19)]

$${}_0\mathcal{F}_1(m;x)=\sum_{k=0}^{\infty}\frac{z^k(m-1)!}{k!(m+k-1)!} \tag{11}$$

and consequently the corresponding definite integral in Table II becomes

$$\mathcal{I}_{i,j}=\sum_{k=0}^{\infty}\left(\frac{(t-s)!\omega_j^k}{k!(t-s+k)!}\right)\Big(\gamma(t-s+k+i,x\lambda_s)-\gamma(t-s+k+i,\lambda_s)\Big). \tag{12}$$

This expression converges quickly for most practical MIMO configurations and K-factors, e.g. $t \leq 20$ and $K_r \leq 10$ dB, and thus can be truncated to a finite number of terms.

V. Exact and Asymptotic Expressions for the SCN CDF/PDF of Dual Wishart Matrices

We now focus on the case of dual Wishart matrices with arbitrary DoF, i.e. $s = 2$ and $t \geq 2$. This scenario is particularly important for MIMO systems, where size limitations of the mobile terminals typically restrict the numbers of antennas. Due to space constraints, we will give results only for dual semi-correlated xentral Wishart matrices[3] for which we derive exact polynomial SCN CDF/PDF expressions and also investigate the probability of experiencing an "extremely large" condition number. This is an important issue for various theoretical and practical problems, such as in predicting the numerical stability of matrix inversion schemes, evaluating the performance of linear MIMO detectors [17], and determining the robustness of coding algorithms [32].

Theorem 2: The SCN CDF of $\mathbf{W} \overset{\mathrm{d}}{\sim} \mathcal{CW}_2(t, \mathbf{\Sigma}_2)$ is

$$F_z(x) = K_{cc}(p_2(x) - p_2(1)) \tag{13}$$

where $p_2(y)$ is defined in (14), with

$$\begin{aligned}&\Delta_3(m, n, \mu, \nu, y)\\ &= (n-1)!\left(\frac{m!}{\mu^n \nu^{m+1}} - \sum_{k=0}^{n-1} \frac{(m+k)! y^k}{k! \mu^{n-k}(\mu y + \nu)^{m+k+1}}\right).\end{aligned} \tag{15}$$

Proof: We start by particularizing (9) to the semi-correlated central Wishart scenario using Table I. Substituting $s = 2$, expanding the determinant and integrating on a term-by-term basis using [28, Eq. (3.351.1)] and [28, Eq. (3.351.3)], we can easily obtain the final result. ■

Corollary 1: The SCN PDF of $\mathbf{W} \overset{\mathrm{d}}{\sim} \mathcal{CW}_2(t, \mathbf{\Sigma}_2)$ is given in (16) at the bottom of the page, where

$$\begin{aligned}&\Delta_4(m, n, \mu, \nu, y)\\ &= -(n-1)! \sum_{k=0}^{n-1} \frac{(m+k)! y^{k-1}(k\nu - \mu(m+1)y)}{k! \mu^{n-k}(\mu y + \nu)^{m+k+2}}.\end{aligned} \tag{17}$$

Proof: Obtained by differentiating (13). ■

[3]The cases of dual uncorrelated central and non-central Wishart matrices are assessed in [29].

An insightful interpretation of the effects of spatial correlation on the channel conditioning is obtained via the following corollary which returns the asymptotic tail distribution for the case of dual semi-correlated Wishart matrices.

Corollary 2: Let z denote the SCN of $\mathbf{W} \overset{\mathrm{d}}{\sim} \mathcal{CW}_2(t, \mathbf{\Sigma}_2)$. Then, as x tends to infinity,

$$\Pr(z > x) \sim \frac{(2t-2)!}{((t-1)!)^2} \frac{\left(\sigma_2^{2t-1} - \sigma_1^{2t-1}\right)}{|\mathbf{\Sigma}_2|^{t-1}(\sigma_2 - \sigma_1) x^{t-1}}, \quad (x \to \infty). \tag{18}$$

Proof: Obtained by taking z large in (16), and integrating. ■

This result shows that the presence of spatial correlation tends to increase the probability of experiencing a highly ill-conditioned channel. To see this, note that the determinant in the denominator of (18) varies inversely with the level of correlation, satisfying $0 \leq |\mathbf{\Sigma}_2| \leq 1$, with the right-hand side equality holding when the channel is uncorrelated (i.e. $\mathbf{\Sigma}_2 = \mathbf{I}_2$). Moreover, the ratio $\left(\sigma_2^{2t-1} - \sigma_1^{2t-1}\right)/(\sigma_2 - \sigma_1)$ varies monotonically with the level of correlation, attaining its minimal value, $2t-1$, when $\sigma_1 = \sigma_2 = 1$.

VI. Numerical Results

In this section, the theoretical results presented in Section IV and Section V are validated via Monte-Carlo simulations. The simulation results are obtained by generating 50,000 random samples of the random matrix $\mathbf{H}$ according to (2)–(4), for the three different Wishart scenarios under investigation. For the semi-correlated Wishart case, we construct $\mathbf{\Sigma}_s$ according to the widely used exponential correlation model, due to its simplicity. Specifically, the entries of $\mathbf{\Sigma}_s$ are generated as $(\mathbf{\Sigma}_s)_{i,j} = \rho^{|i-j|}$, where $\rho \in [0, 1)$.

Figure 1 validates the generic framework for the SCN CDF, as presented in Theorem 1. A 5×3 MIMO channel is considered under all three different fading conditions: uncorrelated, semi-correlated Rayleigh (with $\rho = 0.75$), and uncorrelated Ricean (with $K_r = 3$ dB). For the latter case, the mean channel matrix, $\mathbf{H}_{\mathrm{L}}$, is constructed as full-rank and the eigenvalues of the non-centrality matrix in (5) are $\boldsymbol{\omega} = K_r \cdot [13.56, 1.43, 0.01]^T$, while the infinite series representation of (12) has been truncated to the first 20 terms.

It is clearly that the analytical curves yield an excellent match with the simulator output for all fading scenarios, even

$$\begin{aligned}p_2(y) &= \Delta_3(t-1, t-1, 1/\sigma_1, 1/\sigma_2, y) - \Delta_3(t-2, t, 1/\sigma_1, 1/\sigma_2, y)\\ &\quad - \Delta_3(t-1, t-1, 1/\sigma_2, 1/\sigma_1, y) + \Delta_3(t-2, t, 1/\sigma_2, 1/\sigma_1, y).\end{aligned} \tag{14}$$

$$\begin{aligned}f(z) = K_{cc}\Big(&\Delta_4(t-1, t-1, 1/\sigma_1, 1/\sigma_2, z) - \Delta_4(t-2, t, 1/\sigma_1, 1/\sigma_2, z)\\ &- \Delta_4(t-1, t-1, 1/\sigma_2, 1/\sigma_1, z) + \Delta_4(t-2, t, 1/\sigma_2, 1/\sigma_1, z)\Big), \qquad z \geq 1\end{aligned} \tag{16}$$

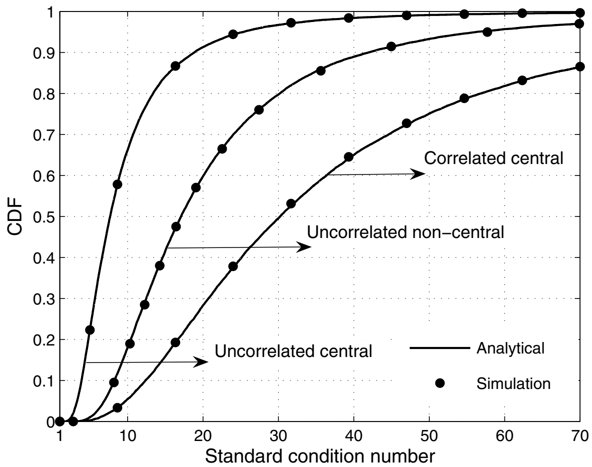

Fig. 1. SCN CDF of a 5×3 MIMO channel under uncorrelated Rayleigh, semi-correlated Rayleigh ($\rho = 0.75$) and uncorrelated Ricean ($K_r = 3$ dB) fading.

with such a low number of terms for the non-central case. Moreover, the presence of either spatial correlation or non-zero mean tends to increase the spread of the SCN distribution and also shift the corresponding CDF to the right, thereby degrading the conditioning of the channel.

In Fig. 2, the effects of spatial correlation on the SCN distribution of dual semi-correlated Wishart matrices are illustrated. The analytical curves have been generated using Theorem 2. Once more, the match between theory and simulation is excellent. More importantly, the figure confirms the common belief that the presence of spatial correlation leads to a significant reduction in the conditioning of the channel; in the MIMO context, this phenomenon corresponds to a reduction in the degree of diversity afforded by the wireless channel.

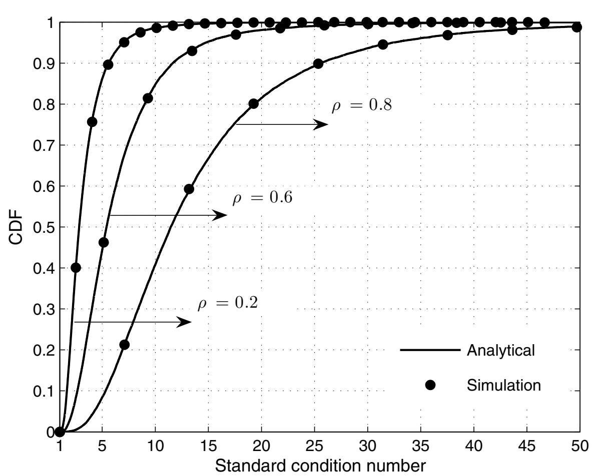

Fig. 2. SCN CDF of a 6×2 semi-correlated Rayleigh MIMO channel for different correlation coefficients, ρ.

Finally, we address the impact of the K-factor on the SCN statistics (c.f. Fig. 3). Note that the structure of the mean channel matrix is the same as before. It is noteworthy that as K_r increases, the mean and the variance of the SCN increase as well, demonstrating that the conditioning of the channel degrades with K_r. Note that this behavior is due to the structure of the channel mean whose subchannels exhibit strong correlation between their LoS rays' phases. In the limit, as $K_r \to \infty$, the SCN will grow infinite.

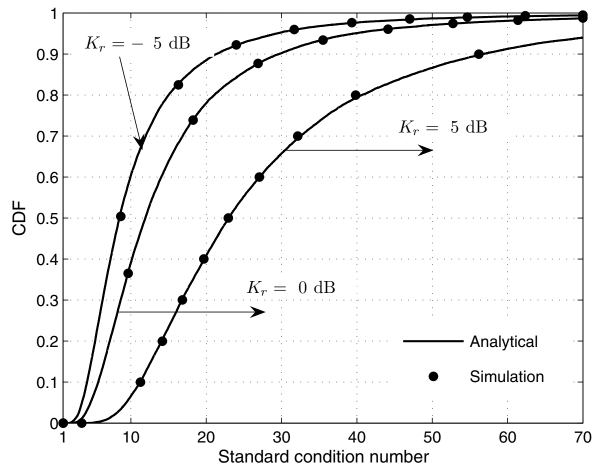

Fig. 3. SCN PDF of a 5×3 uncorrelated Ricean MIMO channel for different K-factors, K_r.

VII. Conclusion

The condition number distribution of complex Wishart matrices is of key importance in the field of MIMO systems, as well as in various other applications. In this contribution, we have presented a novel generic framework for the SCN CDF of three different classes of Wishart matrices, which can inherently describe the following fading scenarios: uncorrelated/semi-correlated Rayleigh and uncorrelated Ricean fading. In addition, polynomial expressions were deduced for the SCN CDF/PDF of dual semi-correlated central Wishart matrices and likewise for the asymptotic tail distribution. The latter result clearly indicates the harmful effects of spatial correlation on channel conditioning. We note that all formulae involved throughout are easily evaluated and efficiently programmed. The implications of the model parameters on the SCN performance were also explored in order to gain fundamental insights into the key parameters affecting channel conditioning.

Appendix A
Proof of Theorem 1

The SCN CDF is evaluated via the following integral

$$F_z(x) = \int_0^{\infty} \left[\int_{\lambda_2}^{x\lambda_s} \cdots \int_{\lambda_s}^{x\lambda_s} f(\boldsymbol{\lambda}) d\lambda_{s-1} \ldots d\lambda_1 \right] d\lambda_s.$$

The multiple integral term inside the square brackets denotes the probability that $(\lambda_1, \lambda_2, \ldots, \lambda_{s-1})$ all lie within the interval $[\lambda_s, x\lambda_s]$. Since $f(\boldsymbol{\lambda})$ is symmetric in all of its arguments, we can reformulate the above equation as

$$F_z(x) = \frac{1}{(s-1)!}\int_0^\infty \int_{\lambda_s}^{x\lambda_s} \cdots \int_{\lambda_s}^{x\lambda_s} f(\boldsymbol{\lambda}) d\lambda_1 \ldots d\lambda_{s-1} d\lambda_s. \tag{19}$$

Now, substituting (6) into (19) and expanding $|\boldsymbol{\Phi}(\boldsymbol{\lambda})|$ using the definition of the determinant (i.e. Liebniz determinant formula), we obtain

$$F_z(x) = \frac{K}{(s-1)!}\int_0^\infty \sum_{\alpha}(-1)^{\alpha} \int_{\lambda_s}^{x\lambda_s} \cdots \int_{\lambda_s}^{x\lambda_s} |\phi_{\alpha_i}(\lambda_i) \times \psi_j(\lambda_i)\xi(\lambda_i)| d\lambda_1 \ldots d\lambda_{s-1} d\lambda_s \tag{20}$$

where $\alpha = \{\alpha_1, \alpha_2, \ldots, \alpha_s\}$ is a permutation of the integers $\{1, 2, \ldots, s\}$, and the summation is taken over all such permutations. This can be further simplified as

$$F_z(x) = \frac{K}{(s-1)!}\int_0^\infty \sum_{\alpha}(-1)^{\alpha} |I_{\alpha_i, j}| d\lambda_s \tag{21}$$

where the elements of the matrix $I_{a_i,j}, (1 \leq i, j \leq s)$, are

$$I_{\alpha_i,j} = \begin{cases} \displaystyle\int_{\lambda_s}^{x\lambda_s} \phi_{\alpha_i}(u)\psi_j(u)\xi(u)du, & \text{if} \quad i \neq s \\ \phi_{\alpha_i}(\lambda_s)\psi_j(\lambda_s)\xi(\lambda_s), & \text{if} \quad i = s. \end{cases} \tag{22}$$

After reordering the rows, $|I_{\alpha_i,j}|$ can be written as

$$|I_{\alpha_i,j}| = (-1)^{\alpha} \left| \begin{bmatrix} \int_{\lambda_s}^{x\lambda_s} \phi_{\alpha_i}(u)\psi_j(u)\xi(u)du, & \text{if } i \neq \alpha_s \\ \phi_{\alpha_i}(\lambda_s)\psi_j(\lambda_s)\xi(\lambda_s), & \text{if } i = \alpha_s \end{bmatrix} \right| \tag{23}$$

The result now follows by substituting (23) into (21) and collecting common terms in the summation.

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