Optimal Transmitters for Control over a Noisy Link with Imperfect Feedback

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***Abstract*— We consider a control loop which is closed over an additive noise communication channel. The quantity of interest is the Signal-to-Noise Ratio (SNR) of the channel such that control objectives like stabilization or minimal tracking error are met. We investigate a transmitter that uses observations of the plant output and the channel output which are fed back to the transmitter. Recently, a transmitter has been proposed which uses linearly filtered channel output feedback and the analytical expression for the minimal value of the SNR has been determined. In the present work, we derive the transmitter in a general framework using noisy channel output feedback. This allows for the investigation of the connection between the quality of imperfect feedback information and performance improvement due to this information. Additionally, other performance criteria than SNR can be used for transmitter design.**

I. Introduction

The investigation of closed loop control systems which are closed over communication channels and the consideration of the limited communication resources like power, rate, bandwidth etc. has attracted considerable attention during the last decades. One assumption that can be observed quite often in the analysis of such systems is a feedback link which provides the transmitter at the input of the channel in the control loop with information about the output of the channel [1]–[5]. In this case, the transmitter has information about the state of the receiver (decoder). This allows for encoding operations that take into account such information like, e. g., an estimate of the system state. Additionally, such information patterns may establish the optimality of the separated design of control and communication. Note that for scenarios considering a discrete but error free (i. e., rate constrained) channel [6]–[9], ideal channel output feedback is always available since transmitted and received signals are identical.

The analysis of rate constrained channels showed that optimal encoding and decoding schemes are in general non-linear and quite complex [9]. This brought up the question how linear transmission schemes together with simple quantizers should be designed in such scenarios and how large the rate loss is compared to optimal schemes [10]. A key step in the analysis provided in [10] was to use and extend results known from the investigation of control systems closed over unquantized, additive noise channels [11]–[13]. It could be shown that a transmitter that performs a linear filtering of the quantization error together with a memoryless entropy coded dithered quantizer is not worse than 1.25 bits per sample compared to the optimal transmission scheme. The linear filter has been calculated using a Gaussian noise model which is one reason for the loss since the actual quantization error has a different distribution. For the Gaussian model, the determination of the minimal rate that ensures the stabilizability of an unstable system relates to the determination of the minimal signal to noise ratio for this goal.

Besides the evaluation of the performance loss w.r.t. the optimal encoding and decoding scheme, the example of [10] shows the importance of the research on linear transmission schemes for control loops that are closed over additive noise channels. The present work revisits the scenario presented in [10] for the additive noise channel and investigates the optimal design of linear transmitters which is generalized to incorporate non-ideal channel output feedback. The main contributions of the paper are summarized below:

- A symmetrical noise scenario for the control and the feedback channel is considered. Here, both the control channel and the feedback channel (which is used to send information from the output of the control channel back to the transmitter at the input) are assumed to be noisy, i. e., non-ideal. This allows to analyze the trade-off between the quality of feedback information and its potential to reduce the negative effect of the noisy control channel.
- The transmitter is determined by solving a generic optimization problem. The optimization w.r.t. a communication objective like the Signal-to-Noise Ratio (SNR) turns out to be a Linear Quadratic Gaussian (LQG) control problem. Thus, a larger class of control and communication objectives can be considered for transmitter optimization and optimality is established beyond asymptotic results.
- The expressions for the optimal transmitter allow for a better understanding of the actual transmit operations. In [10], the transmitter is designed for a control loop with a predetermined controller for a nominal system behavior. With the presented approach, it is possible to show that the optimal transmitter opens this control loop and closes it again with a control signal that is optimal w.r.t. the communication objective for the open loop interconnection of the predesigned controller and the system to be controlled.

Notation: Vectors and matrices are denoted by lower and upper case bold letters (e. g., $\boldsymbol{a}$ and $\boldsymbol{A}$), whereas scalars are lower case letters (e. g., a). The operators $\mathrm{E}[\bullet]$, $\mathrm{E}[\bullet|\,\boldsymbol{a}]$, $(\bullet)^{\mathrm{T}}$, and $\mathrm{tr}[\bullet]$ are expectation, expectation conditioned on the vector $\boldsymbol{a}$, transpose and trace of a matrix, respectively. $\mathbf{I}_N$ is the $N\times N$ identity matrix. The N-dimensional all-zeros vector is denoted by $\mathbf{0}_N$, the $N\times M$ all-zeros matrix by $\mathbf{0}_{N\times M}$. $\mathcal{N}(\boldsymbol{\mu}, \boldsymbol{C}_{\boldsymbol{a}})$ denotes the Gaussian distribution of the real random vector $\boldsymbol{a}$ with mean $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{C}_{\boldsymbol{a}} = \mathrm{E}\left[(\boldsymbol{a}-\boldsymbol{\mu})(\boldsymbol{a}-\boldsymbol{\mu})^{\mathrm{T}}\right]$.

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II. System and Channel Model

A. System Model

We consider a control loop that is closed over one communication channel which is located between the output of the plant G to be controlled and the input of a predesigned controller C. It is not taken into account how the controller has been designed.

The external signals acting on the control loop are the plant process noise $\boldsymbol{w}_k$, the observation noise v_k and the reference signal r_k. The transmitter is located at the output of the plant and generates the channel input signal t_k given the observations of the system output and the one step delayed channel output η_{k-1}, which is fed back to the transmitter using a second channel. The overall system is depicted in Fig.1.

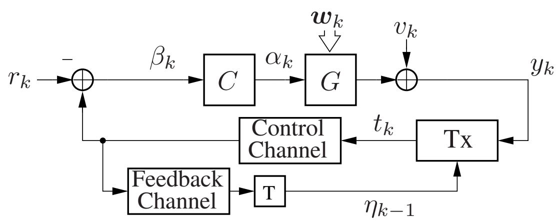

Fig. 1. Control loop closed over a communication channel with transmitter using causal channel output feedback.

We assume that both the plant G and the controller C are linear, time invariant, discrete time, single-input single-output[1] systems with state space representation

$$\begin{aligned} \boldsymbol{x}_{k+1}^{(G)} &= \boldsymbol{A}_G \boldsymbol{x}_k^{(G)} + \boldsymbol{b}_G \alpha_k + \boldsymbol{w}_k \\ y_k &= \boldsymbol{c}_G^{\mathrm{T}} \boldsymbol{x}_k^{(G)} + v_k \end{aligned} \tag{1}$$

and

$$\begin{aligned} \boldsymbol{x}_{k+1}^{(C)} &= \boldsymbol{A}_C \boldsymbol{x}_k^{(C)} + \boldsymbol{b}_C \beta_k \\ \alpha_k &= \boldsymbol{c}_C^{\mathrm{T}} \boldsymbol{x}_k^{(C)} + d_C \beta_k, \end{aligned} \tag{2}$$

with system states $\boldsymbol{x}_k^{(G)} \in \mathbb{R}^N$ and $\boldsymbol{x}_k^{(C)} \in \mathbb{R}^M$, system matrices $\boldsymbol{A}_G \in \mathbb{R}^{N\times N}$ and $\boldsymbol{A}_C \in \mathbb{R}^{M\times M}$, system input vectors $\boldsymbol{b}_G \in \mathbb{R}^N$ and $\boldsymbol{b}_C \in \mathbb{R}^M$ and system output vectors $\boldsymbol{c}_G \in \mathbb{R}^N$ and $\boldsymbol{c}_C \in \mathbb{R}^M$, respectively, and $d_C \in \mathbb{R}$. The input signals of G and C are α_k and β_k, respectively.

For the sake of simplicity, we assume that $\boldsymbol{w}_k \in \mathbb{R}^N$, v_k and r_k are mutually independent i. i. d. sequences with respective distributions $\mathcal{N}(\boldsymbol{0}_N, \boldsymbol{C}_{\boldsymbol{w}})$, $\mathcal{N}(0, c_v)$ and $\mathcal{N}(0, c_r)$. It is straight forward to consider correlated processes which allow for a state space representation. In this case, the results presented in Sec. III remain the same, while for the optimality of the transmitter, the process models have to be taken into account.[2]

B. Channel Model and Transmitter

The model of the channel that will be used is the standard Additive White Gaussian Noise (AWGN) channel, i. e., for an input symbol s_k, the channel output z_k reads as

$$z_k = s_k + n_k, \tag{3}$$

where $n_k \sim \mathcal{N}(0, c_n)$ is an i.i.d. sequence which is mutually independent of all other external signals. The AWGN model will be applied to both channels in Fig. 1.

The system model which will be used in the following is depicted in Fig. 2.

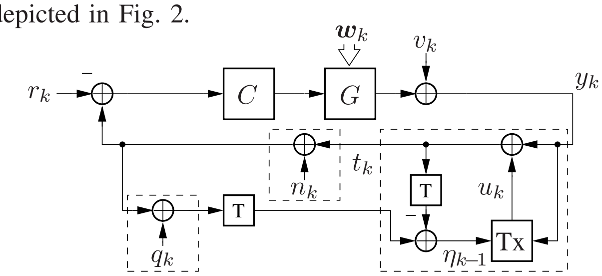

Fig. 2. Closed loop system with channel output feedback.

We assume that the transmitter generates an additive[3] signal u_k for the plant output which results in the channel input sequence t_k. The information for the generation of u_k is the channel output feedback η_{k-1} and the observation of the plant output y_k. In order to keep calculations simple, we subtract the known part t_{k-1} from the channel output feedback in order to generate η_{k-1}, i. e.,

$$\eta_{k-1} = n_{k-1} + q_{k-1}, \tag{4}$$

where the channel noise sequences $n_k \sim \mathcal{N}(0, c_n)$ and $q_k \sim \mathcal{N}(0, c_q)$ are assumed to be mutually independent i.i.d. sequences and independent of all other random sequences according to the AWGN channel model. The subtraction of t_{k-1} does not affect the result for the optimal transmitter. We do not restrict the memory of the transmitter, i. e., at time k, we assume the knowledge of

$$\boldsymbol{\phi}_k = [y_0, y_1, \ldots, y_k, \eta_0, \eta_1, \ldots, \eta_{k-1}, u_0, u_1, \ldots, u_{k-1}]^{\mathrm{T}}. \tag{5}$$

In Sec. III-A, it will be shown that this is not an idealization and can be implemented with finite memory requirements.

In order to determine the transmit signal t_k, we set up the state space model of the feedback interconnection of G and C with external inputs r_k, $\boldsymbol{w}_k$, v_k and n_k:[4]

$$\begin{aligned} \boldsymbol{\zeta}_{k+1} &= \boldsymbol{A}\boldsymbol{\zeta}_k + \boldsymbol{b}(u_k + n_k - r_k) + \boldsymbol{S}^{\mathrm{T}} \boldsymbol{w}_k + \mathbf{e} v_{k+1} \\ y_k &= \boldsymbol{c}^{\mathrm{T}} \boldsymbol{\zeta}_k, \end{aligned} \tag{6}$$

with system state $\boldsymbol{\zeta}_k = \left[\boldsymbol{x}_k^{(G),\mathrm{T}}, \boldsymbol{x}_k^{(C),\mathrm{T}}, v_k\right]^{\mathrm{T}} \in \mathbb{R}^{N+M+1}$. The corresponding system matrix of appropriate dimension is

$$\boldsymbol{A} = \begin{bmatrix} \boldsymbol{A}_G + \boldsymbol{b}_G \boldsymbol{c}_G^{\mathrm{T}} d_C & \boldsymbol{b}_G \boldsymbol{c}_C^{\mathrm{T}} & \boldsymbol{b}_G d_C \\ \boldsymbol{b}_C \boldsymbol{c}_G^{\mathrm{T}} & \boldsymbol{A}_C & \boldsymbol{b}_C \\ \boldsymbol{0}_N^{\mathrm{T}} & \boldsymbol{0}_M^{\mathrm{T}} & 0 \end{bmatrix} \tag{7}$$

and the system input and output vectors are

$$\begin{aligned} \boldsymbol{b} &= \left[d_C \boldsymbol{b}_G^{\mathrm{T}}, \boldsymbol{b}_C^{\mathrm{T}}, 0\right]^{\mathrm{T}} \in \mathbb{R}^{N+M+1}, \\ \boldsymbol{c} &= \left[\boldsymbol{c}_G^{\mathrm{T}}, \boldsymbol{0}_M^{\mathrm{T}}, 1\right]^{\mathrm{T}} \in \mathbb{R}^{N+M+1}, \end{aligned} \tag{8}$$

[1] Almost all presented results can readily be extended to MIMO systems, except for the expression for the minimal SNR shown in Theorem 1.

[2] This is accomplished by state augmentation, see, e. g., [14].

[3] Note that this is no restriction since any transmitter Tx could use the knowledge of y_k to subtract it, effectively setting $t_k = u_k$ as the channel input. We will see that the optimal transmitter operates exactly this way.

[4] Comparing with Eq. (2), we observe that $\beta_k = y_k + u_k + n_k - r_k$.

respectively. Finally, $\boldsymbol{S} = \left[\mathbf{I}_N, \mathbf{0}_{N\times(M+1)}\right]$ and $\mathbf{e}$ is the last column of $\mathbf{I}_{N+M+1}$.

Note that we included the observation noise v_k in the system state $\boldsymbol{\zeta}_k$ despite the fact that it is assumed to be an i.i.d. sequence. The reason for this step is that it simplifies the expressions for the optimal state estimator presented in Appendix A and the cost function in Sec. III-A.

III. Optimization of the Transmitter

As a generic problem of transmitter optimization, we consider the minimization of the transmit power, i. e., the variance of the transmit signal t_k, in Sec. III-A. The expression for the associated SNR is derived in Sec. III-B. In Sec. III-C the result will be extended to include control specific objectives in the optimization of the transmitter.

A. Minimization of Transmit Power

For the determination of the optimal transmitter, we assume that the feedback loop can be stabilized with a time invariant controller and transmitter, which ensures that the distributions of all random variables converge exponentially fast to the respective stationary distributions for any distribution of initial values. Thus, we use the following definition:

Definition 1 (Variance)**.** *For an asymptotically stationary random sequence t_k with expected value $\mathrm{E}[t_k] = 0, \forall k$, the variance is given by*

$$c_t = \lim_{k\to\infty} \mathrm{E}\left[t_k^2\right]. \tag{9}$$

Since the power of the transmit signal t_k is considered, together with Definition 1, the cost function to be minimized w. r. t. the additive signal u_k reads as

$$J = c_t. \tag{10}$$

Due to the fact that the distributions of all random variables converge to stationary distributions and by using the equality

$$t_k^2 = (y_k + u_k)^2 = \boldsymbol{\zeta}_k^{\mathrm{T}} \boldsymbol{c}\boldsymbol{c}^{\mathrm{T}} \boldsymbol{\zeta}_k + u_k^2 + 2\boldsymbol{\zeta}_k^{\mathrm{T}} \boldsymbol{c} u_k, \tag{11}$$

the cost function can be rewritten in the well known form

$$J = \lim_{K\to\infty} \frac{1}{K} \mathrm{E}\left[(\boldsymbol{\zeta}_K^{\mathrm{T}}\boldsymbol{c})^2 + \sum_{k=1}^{K-1} \boldsymbol{\zeta}_k^{\mathrm{T}} \boldsymbol{c}\boldsymbol{c}^{\mathrm{T}} \boldsymbol{\zeta}_k + u_k^2 + 2\boldsymbol{\zeta}_k^{\mathrm{T}} \boldsymbol{c} u_k\right], \tag{12}$$

which allows for the application of standard results from LQG control. The optimization problem thus reads as

$$\underset{u_k,\, k=0,1,\ldots}{\text{minimize}} \; J \quad \text{s. t.} \quad u_k = \rho_k\left(\boldsymbol{\phi}_k\right), \tag{13}$$

i. e., the additive signal u_k is constrained to be a function $\rho_k(\boldsymbol{\phi}_k)$ of the information $\boldsymbol{\phi}_k$ which is available to the transmitter at time k.

Since the system and the channels are linear and all external distortions are assumed to be white Gaussian sequences, the solution of the optimization problem is the well known LQG controller [15, p. 294],

$$u_k = \boldsymbol{l}^{\mathrm{T}} \hat{\boldsymbol{\zeta}}_k, \tag{14}$$

with the control vector

$$\boldsymbol{l} = -\left(\boldsymbol{A}^{\mathrm{T}}\boldsymbol{K}\boldsymbol{b} + \boldsymbol{c}\right)\left(\boldsymbol{b}^{\mathrm{T}}\boldsymbol{K}\boldsymbol{b} + 1\right)^{-1}, \tag{15}$$

where $\boldsymbol{K}$ is the solution of the Discrete Algebraic Riccati Equation (DARE)

$$\begin{aligned}\boldsymbol{K} = {} & \boldsymbol{A}^{\mathrm{T}}\boldsymbol{K}\boldsymbol{A} + \boldsymbol{c}\boldsymbol{c}^{\mathrm{T}} \\ & - \left(\boldsymbol{A}^{\mathrm{T}}\boldsymbol{K}\boldsymbol{b} + \boldsymbol{c}\right)\left(\boldsymbol{b}^{\mathrm{T}}\boldsymbol{K}\boldsymbol{b} + 1\right)^{-1}\left(\boldsymbol{A}^{\mathrm{T}}\boldsymbol{K}\boldsymbol{b} + \boldsymbol{c}\right)^{\mathrm{T}}\end{aligned} \tag{16}$$

and the optimal estimate of the system state

$$\hat{\boldsymbol{\zeta}}_k = \mathrm{E}\left[\boldsymbol{\zeta}_k \middle| \boldsymbol{\phi}_k\right]. \tag{17}$$

With this result, the minimal cost, i. e., the minimal transmit power required for a bounded variance of the system states, is given by [15, p. 295]

$$\begin{aligned}J^* &= \underset{u_k, k=0,1,\ldots}{\operatorname{argmin}} \; J \\ &= \operatorname{tr}\left[\boldsymbol{K}\left(\boldsymbol{b}\boldsymbol{b}^{\mathrm{T}}(c_n + c_r) + \boldsymbol{S}^{\mathrm{T}}\boldsymbol{C}_{\boldsymbol{w}}\boldsymbol{S} + \mathbf{e}\mathbf{e}^{\mathrm{T}}c_v\right) + \boldsymbol{P}\boldsymbol{C}_{\tilde{\boldsymbol{\zeta}}}\right],\end{aligned} \tag{18}$$

where $\boldsymbol{P} = \boldsymbol{A}^{\mathrm{T}}\boldsymbol{K}\boldsymbol{A} - \boldsymbol{K} + \boldsymbol{c}\boldsymbol{c}^{\mathrm{T}}$ and $\boldsymbol{C}_{\tilde{\boldsymbol{\zeta}}}$ is the covariance matrix of the estimation error $\tilde{\boldsymbol{\zeta}}_k = \boldsymbol{\zeta}_k - \hat{\boldsymbol{\zeta}}_k$.

The computation of the optimal estimate $\hat{\boldsymbol{\zeta}}_k$ is realized by the Kalman filter. This shows that the assumption of an unrestricted memory size of the transmitter (cf. Eq. 5) does not prohibit the practical applicability of the presented approach. Note that due to the observation noise q_k in the feedback channel (cf. Fig. 2), it is necessary to modify the Kalman filter algorithm, which is shown in Appendix A.

B. Minimal Value of the SNR

Since any input signal of an AWGN channel can be scaled to have a desired power and restored at the output using the inverse of the scaling factor, the parameter which describes the quality of such a channel is not the power of the input signal but the SNR. We will use the following definition:

Definition 2 (SNR)**.** *Using the transmitter given by Eq. (14), the SNR of the control channel is*

$$\gamma = \frac{J^*}{c_n}, \tag{19}$$

where J^* is given by Eq. (18).

The infimal value of the SNR which allows for the stabilization of an unstable system has been explored extensively in the past (e. g., [10]–[12], [16], [17]) and is known to be a function of the unstable eigenvalues of the system to be controlled. It has also been shown that this value can be approached arbitrarily close by increasing the variance c_n such that the channel noise becomes the dominant disturbance. This can be achieved if there is the degree of freedom to choose the variance of the channel noise, e. g., if it is a free system parameter [10] or can be modified by a scaling of the channel input and output signal [17].

Although the following result is not new, it is shown here since it is derived using a different approach compared to the existing ones and demonstrates its ability to reproduce the expressions for the ultimate bound of stabilizability.

Theorem 1. *Consider the system shown in Fig. 2 together with the transmitter given by Eq. (14). Then, for $c_n \to \infty$, the SNR γ approaches the infimal value*

$$\gamma_{\text{inf}} = \boldsymbol{b}^{\mathrm{T}}\boldsymbol{K}\boldsymbol{b} = \Big(\prod\nolimits_{|\lambda_{G,i}|>1}|\lambda_{G,i}|^2\Big)\Big(\prod\nolimits_{|\lambda_{C,i}|>1}|\lambda_{C,i}|^2\Big) - 1, \tag{20}$$

arbitrarily close. Here, $\lambda_{G,i}$, $i \in \{1, 2, \ldots, N\}$, and $\lambda_{C,i}$, $i \in \{1, 2, \ldots, M\}$, are the eigenvalues of $\boldsymbol{A}_G$ and $\boldsymbol{A}_C$, respectively.

Proof. Inserting the expression for J^* given by Eq. (18) in Eq. (19), we find that the constant terms depending on c_r, $\boldsymbol{C_w}$ and c_v converge to zero for $c_n \to \infty$. Additionally, the error covariance matrix $\boldsymbol{C}_{\tilde{\boldsymbol{\zeta}}}$ converges to a constant matrix (cf Eq. 33). Thus, the contribution of the estimation error to the SNR converges to zero. Consequently, the infimal value of the SNR is given by

$$\gamma_{\text{inf}} = \lim_{c_n \to \infty} \gamma = \boldsymbol{b}^{\mathrm{T}}\boldsymbol{K}\boldsymbol{b}. \tag{21}$$

Next, we introduce the system matrix of the open loop interconnection of G and C,

$$\boldsymbol{A}_{\text{OL}} = \boldsymbol{A} - \boldsymbol{b}\boldsymbol{c}^{\mathrm{T}} = \begin{bmatrix} \boldsymbol{A}_G & \boldsymbol{b}_G\boldsymbol{c}_C^{\mathrm{T}} & \boldsymbol{0}_N \\ \boldsymbol{0}_{M\times N} & \boldsymbol{A}_C & \boldsymbol{0}_M \\ \boldsymbol{0}_N^{\mathrm{T}} & \boldsymbol{0}_M^{\mathrm{T}} & 0 \end{bmatrix}. \tag{22}$$

With $\boldsymbol{A}_{\text{OL}}$, the solution $\boldsymbol{K}$ of the DARE (16) is also the solution of (cf. [18])

$$\boldsymbol{K} = \boldsymbol{A}_{\text{OL}}^{\mathrm{T}}\left(\boldsymbol{K} - \boldsymbol{K}\boldsymbol{b}\left(\boldsymbol{b}^{\mathrm{T}}\boldsymbol{K}\boldsymbol{b}+1\right)^{-1}\boldsymbol{b}^{\mathrm{T}}\boldsymbol{K}\right)\boldsymbol{A}_{\text{OL}}. \tag{23}$$

This matrix corresponds to an expensive control LQR problem which is known to have the property (e. g., [7], [17])

$$\boldsymbol{b}^{\mathrm{T}}\boldsymbol{K}\boldsymbol{b} = \left(\prod\nolimits_{|\lambda_i|>1}|\lambda_i|^2\right) - 1, \tag{24}$$

where λ_i, $i \in \{1, 2, \ldots, N+M+1\}$, are the eigenvalues of $\boldsymbol{A}_{\text{OL}}$. Noting that the set of unstable eigenvalues of $\boldsymbol{A}_{\text{OL}}$ is the union of the set of unstable eigenvalues of $\boldsymbol{A}_G$ and $\boldsymbol{A}_C$ since $\boldsymbol{A}_{\text{OL}}$ is block triangular, Eq. (20) follows. □

C. General Transmitter Optimization

With the interpretation of the transmitter optimization as an LQG control problem, it is natural to include additional performance requirements and transmit power constraints.

For the consideration of further performance criteria, a positive semidefinite weight matrix $\boldsymbol{Q}$ is introduced (cf. [15]). The resulting cost function reads as (cf. Eq. 12)

$$J_{\text{LQG}} = \lim_{K\to\infty}\frac{1}{K}\,\mathrm{E}\left[\boldsymbol{\zeta}_K^{\mathrm{T}}\boldsymbol{Q}\boldsymbol{\zeta}_K + \sum_{k=0}^{K-1}\boldsymbol{\zeta}_k^{\mathrm{T}}\boldsymbol{Q}\boldsymbol{\zeta}_k + rt_k^2\right], \tag{25}$$

with t_k^2 given by Eq. (11). Note that we also introduced a weighting factor $r \geq 0$ for the transmit signal. This factor can be interpreted in two ways. As a free parameter, it provides the system designer with the possibility to trade off the control objective against the transmit power which is necessary to achieve this objective. If $\boldsymbol{Q}$ is the zero matrix and $r = 1$, we get the cost function of Sec. III-A.

If the minimization of J_{LQG} is additionally subject to a hard limit γ_{limit} for the control channel SNR, i. e.,

$$\underset{u_k,\,k=0,1,\ldots}{\text{minimize}}\;\; J_{\text{LQG}} \quad \text{s. t.} \quad u_k = \rho_k(\boldsymbol{\phi}_k), \quad \frac{c_t}{c_n} \leq \gamma_{\text{limit}}, \tag{26}$$

the weighting factor r of the transmit signal can be associated with a Lagrangian multiplier which establishes the trade-off between the cost function and the constraint such that the latter is fulfilled. We do not go into detail how to solve the above optimization problem but refer the reader to the corresponding literature, e. g., [19], [20].

Irrespective of the consideration of an SNR constraint, the optimal transmitter has a form analogous to Eq. (14) with[5]

$$\boldsymbol{l} = -\left(\boldsymbol{A}^{\mathrm{T}}\boldsymbol{K}\boldsymbol{b} + \nu\boldsymbol{c}\right)\left(\boldsymbol{b}^{\mathrm{T}}\boldsymbol{K}\boldsymbol{b} + \nu\right)^{-1} \tag{27}$$

and (cf. Eq. 23 with $\boldsymbol{Q} = \boldsymbol{0}_{(N+M+1)\times(N+M+1)}$ and $\nu = 1$)

$$\boldsymbol{K} = \boldsymbol{A}_{\text{OL}}^{\mathrm{T}}\Big(\boldsymbol{K} - \boldsymbol{K}\boldsymbol{b}\left(\boldsymbol{b}^{\mathrm{T}}\boldsymbol{K}\boldsymbol{b}+\nu\right)^{-1}\boldsymbol{b}^{\mathrm{T}}\boldsymbol{K}\Big)\boldsymbol{A}_{\text{OL}} + \boldsymbol{Q}. \tag{28}$$

Using the structure of the solution, we can analyze the effect of the transmitter on the transmit signal in more detail.

IV. Properties of the Optimal Transmitter

In Sec. II, we introduced the predesigned controller C without making a statement about the design criteria for this controller. Nevertheless, the following result shows that the optimal transmitter determined according to (26) may have a destructive effect on the predesigned (nominal) control loop.

Theorem 2. *For the system shown in Fig. 2, the optimal transmitter (cf. Eq. 14) which is the solution of the LQG control problem (26) breaks the control loop by subtracting the plant output y_k and replacing it with the optimal control sequence for the open loop interconnection of G and C.*

Proof. Using the matrix $\boldsymbol{A}_{\text{OL}}$ given by Eq. (22), the optimal control vector $\boldsymbol{l}$ (cf. Eq. 27) can be rewritten as

$$\boldsymbol{l} = \boldsymbol{l}_{\text{OL}} - \boldsymbol{c}, \tag{29}$$

where $\boldsymbol{l}_{\text{OL}} = -\boldsymbol{A}_{\text{OL}}^{\mathrm{T}}\boldsymbol{K}\boldsymbol{b}\left(\boldsymbol{b}^{\mathrm{T}}\boldsymbol{K}\boldsymbol{b}+\nu\right)^{-1}$. With this expression for $\boldsymbol{l}$ and Eq. (14) the transmit signal t_k reads as

$$t_k = y_k + u_k = \boldsymbol{c}^{\mathrm{T}}\boldsymbol{\zeta}_k + \boldsymbol{l}_{\text{OL}}^{\mathrm{T}}\hat{\boldsymbol{\zeta}}_k - \boldsymbol{c}^{\mathrm{T}}\hat{\boldsymbol{\zeta}}_k = \boldsymbol{l}_{\text{OL}}^{\mathrm{T}}\hat{\boldsymbol{\zeta}}_k, \tag{30}$$

because $\boldsymbol{c}^{\mathrm{T}}\tilde{\boldsymbol{\zeta}}_k = 0$ due to the ideal observation of $y_k = \boldsymbol{c}^{\mathrm{T}}\boldsymbol{\zeta}_k$ at the plant output.[6] Since the observation $\boldsymbol{c}^{\mathrm{T}}\boldsymbol{\zeta}_k = \boldsymbol{c}^{\mathrm{T}}\hat{\boldsymbol{\zeta}}_k$ is subtracted by the transmitter, the remaining signal $\boldsymbol{l}_{\text{OL}}^{\mathrm{T}}\hat{\boldsymbol{\zeta}}_k$ has to be a stabilizing controller for the open loop interconnected system of G and C. This is true since $\boldsymbol{l}_{\text{OL}}$ corresponds to the optimal regulator for this system and $\hat{\boldsymbol{\zeta}}_k$ is the optimal state estimate which together is the optimal LQG controller. □

We see that the optimal transmitter does not take into account the objectives the nominal controller C has been designed for. Instead, it is a controller for the open loop

[5] If no constraint is considered, we have $\nu = r$. With an SNR constraint, ν is determined by the optimal value of the associated dual variable.

[6] With Eq. (33) it can be verified that the variance of $\boldsymbol{c}^{\mathrm{T}}\tilde{\boldsymbol{\zeta}}_k$ is zero.

interconnection of G and C which results in a closed loop performance that is strongly determined by the design criteria for the transmitter. As a consequence, a transmitter which is designed for a minimal SNR may lead to poor closed loop performance w.r.t. the nominal control objective if this is in conflict with the minimization of the SNR.

V. Numerical Example

In this section, we demonstrate two aspects for the proposed transmitter design. The first one is the effect of the noise in the channel output feedback which reduces the usefulness of this information for the transmitter. The second one is the impact of the optimal transmitter on the tracking error variance for a reference tracking problem.

The parameters for the system G and the reference R are taken from [21, chap. 4.6] and read as

$$\boldsymbol{A}_G = \begin{bmatrix} 2 & 0 \\ 1 & 0 \end{bmatrix}, \quad \boldsymbol{b}_G = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \boldsymbol{c}_G = \begin{bmatrix} 1 \\ 0.8 \end{bmatrix} \quad \text{and}$$

$$a_R = 0.9, \;\; b_R = 0.25, \;\; c_R = 0.36, \;\; d_R = 0.1,$$

respectively. The reference model

$$x_{k+1}^{(R)} = a_R x_k^{(R)} + b_R \delta_k,$$
$$r_k = c_R x_k^{(R)} + d_R \delta_k.$$

is driven by the i.i.d. noise sequence $\delta_k \sim \mathcal{N}(0,1)$. As in [21], we chose $\boldsymbol{C_w} = \boldsymbol{0}_{2\times 2}$, $c_v = 0$. The variance of the feedback noise is chosen from the interval $c_q \in [10^{-6}, 10^4]$, that of the control channel is $c_n = 5\cdot 10^{-3}$. Thus, we can not expect to achieve an SNR close to the infimal value of $\gamma_{\text{inf}} = 3$.

Two transmitters are designed according to different objectives. The first one is the minimal transmit power (cf. Sec. III-A), the second one is the minimal tracking error variance which is given by $\lim_{k\to\infty} \mathrm{E}\left[(y_k - r_k)^2\right]$ (cf. Sec. III-C). The controller C is chosen to be the optimal (minimum error variance) LQG tracker which assumes that $t_k = y_k$ and treats the channel noise n_k as additional observation noise. Thus, the channel noise is taken into account also for the nominal design without a transmitter to show the *additional* benefit due to the transmitter.

In order to investigate the effect of the feedback channel noise q_k, we calculate the SNR of the control channel, given by $\frac{c_t}{c_n}$, as well as the SNR of the feedback channel, given by $\frac{c_t + c_n}{c_q}$. The determination of the transmit power $c_t = \boldsymbol{l}_{\text{OL}}^{\text{T}} \boldsymbol{C}_{\hat{\zeta}} \boldsymbol{l}_{\text{OL}}$ is presented in Appendix B.

Fig. 3 shows the variation of the control channel SNR for increasing SNR of the feedback channel. We observe that irrespective of the design criterion for the transmitter, the control channel SNR decreases for increasing quality of the feedback information and there is no benefit in the availability of channel output feedback for a feedback SNR below 0dB. On the other hand, an SNR of 30dB suffices to obtain almost the full reduction of transmit power. It can also be seen that the transmitter designed for minimal tracking error uses slightly more power compared to the case with no transmitter, but can utilize the channel output feedback to get below this reference value of the SNR.

An interesting point is the impact of the transmitter on the tracking error variance which is shown in Fig. 4. As expected, the transmitter designed for minimal transmit power reduces the control channel SNR at the expense of a strongly increased tracking error variance. Additionally, the error increases further when the transmitter can exploit the channel output feedback. Contrary, the transmitter designed for minimal tracking error achieves an error variance which is always below the case with no transmitter. The channel output feedback can be used for a very small decrease of the error variance together with a significant decrease of transmit power. Thus, a system using this transmitter outperforms a system without transmitter in both tracking error and transmit power if high quality channel output feedback is available.

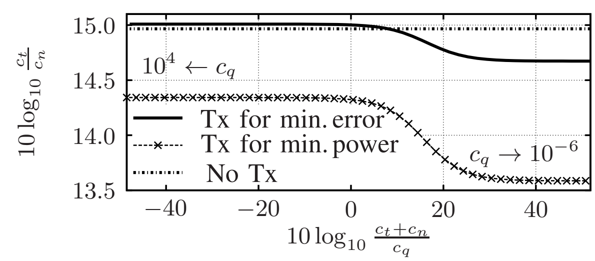

Fig. 3. SNR of the control channel vs. SNR of the feedback channel.

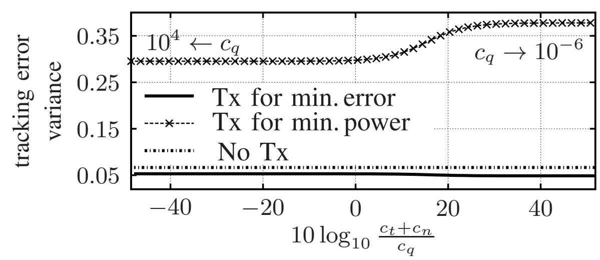

Fig. 4. Variance of the tracking error vs. SNR of the feedback channel.

VI. Discussion

The authors of [10] considered a discrete but error free control channel. In this case, the transmitter has perfect knowledge of the noise introduced by the quantizer at the transmitter which corresponds to a noise free feedback link. The approach in the present work allows for a transmitter design in a scenario where this error free control channel can not be provided. For such a scenario, one can resort to an analog transmission using an AWGN channel. Consequently, the feedback channel for the control channel output is also an AWGN channel. It is now possible for the designer of a control loop to decide if a feedback channel is available such that the quality of the channel output feedback is beneficial for the performance of the overall system.

The presented approach offers the possibility to analyze the properties of the optimal transmitter and to incorporate performance criteria beyond the control channel SNR. For the minimization of the SNR, it could be shown theoretically and numerically that the corresponding optimal transmitter may destroy the control loop behavior and uses the additional channel output feedback information for an SNR reduction at the expense of, e. g., tracking performance. Thus, performance criteria must also be taken into account for the transmitter design, leading to a joint optimization of the transmitter and the controller.

For the determination of ultimate rate or SNR bounds and of transmitters which achieve these bounds, it is of course not possible to take into account performance. But if there is the degree of freedom to operate far away from these bounds, the presented approach provides a tool for optimal transmitter design w.r.t. a larger class of optimality criteria.

VII. Conclusion

We showed how to design the optimal transmitter for a control loop which is closed over an AWGN channel when there is, possibly noisy, channel output feedback available at the transmitter. For perfect channel output feedback, the solution coincides with the one presented in [10]. With the presented approach, the LQG framework could be used to consider also noisy feedback information and performance criteria beyond the channel SNR for the transmitter design.

Furthermore, the effect of the optimal transmitter on the control loop has been analyzed. Since the transmitter breaks the control loop and inserts its own control signal, the choice of the optimality criterion for transmitter design is essential for the overall controller performance. Thus, a further investigation of the joint design of controller and transmitter, although possibly non-convex, seems to be important.

Appendix

A. Kalman Filter with Noisy Input

The two basic operations of the Kalman filter algorithm are prediction and correction. In the prediction step, we compute

$$\begin{aligned}\hat{\boldsymbol{\zeta}}^{\mathrm{P}}_{k+1} &= \mathrm{E}\left[\boldsymbol{\zeta}_{k+1} \,|\, \boldsymbol{\phi}_k, u_k, \eta_k\right] = \boldsymbol{A}\hat{\boldsymbol{\zeta}}_k + \boldsymbol{b}\left(u_k + \mathrm{E}\left[n_k \,|\, \eta_k\right]\right) \\ &= \boldsymbol{A}\hat{\boldsymbol{\zeta}}_k + \boldsymbol{b}\left(u_k + \frac{c_n}{c_n + c_q}\eta_k\right),\end{aligned} \tag{31}$$

since $\eta_k = n_k + q_k$ (cf. Eq. 4). Note that for $c_q = 0$, the above expression reduces to the standard Kalman filter prediction step. The correction step remains unchanged and is given by

$$\hat{\boldsymbol{\zeta}}_k = \hat{\boldsymbol{\zeta}}^{\mathrm{P}}_k + \boldsymbol{g}\left(y_k - \boldsymbol{c}^{\mathrm{T}}\hat{\boldsymbol{\zeta}}^{\mathrm{P}}_k\right), \tag{32}$$

with the Kalman gain $\boldsymbol{g} = \boldsymbol{C}^{\mathrm{P}}_{\tilde{\zeta}}\boldsymbol{c}\left(\boldsymbol{c}^{\mathrm{T}}\boldsymbol{C}^{\mathrm{P}}_{\tilde{\zeta}}\boldsymbol{c}\right)^{-1}$. The estimation error covariance matrix of the prediction step $\boldsymbol{C}^{\mathrm{P}}_{\tilde{\zeta}}$ is modified according to Eq. (31). The error covariance matrix $\boldsymbol{C}_{\tilde{\zeta}}$ of the correction step has the well known expression:

$$\begin{aligned}\boldsymbol{C}_{\tilde{\zeta}} &= \boldsymbol{C}^{\mathrm{P}}_{\tilde{\zeta}} - \boldsymbol{C}^{\mathrm{P}}_{\tilde{\zeta}}\boldsymbol{c}\left(\boldsymbol{c}^{\mathrm{T}}\boldsymbol{C}^{\mathrm{P}}_{\tilde{\zeta}}\boldsymbol{c}\right)^{-1}\boldsymbol{c}^{\mathrm{T}}\boldsymbol{C}^{\mathrm{P}}_{\tilde{\zeta}}, \qquad (33)\\ \boldsymbol{C}^{\mathrm{P}}_{\tilde{\zeta}} &= \boldsymbol{A}\boldsymbol{C}_{\tilde{\zeta}}\boldsymbol{A}^{\mathrm{T}} + \boldsymbol{b}\boldsymbol{b}^{\mathrm{T}}\left(\frac{c_n c_q}{c_n + c_q} + c_r\right) + \boldsymbol{S}^{\mathrm{T}}\boldsymbol{C}_{\boldsymbol{w}}\boldsymbol{S} + \mathbf{e}\mathbf{e}^{\mathrm{T}}c_v.\end{aligned}$$

B. Variance of the Transmit Signal

Inserting the solution of the optimal transmitter (cf. Eq. 14) in the state equation (6), we get

$$\begin{aligned}\boldsymbol{\zeta}_{k+1} &= \boldsymbol{A}\boldsymbol{\zeta}_k + \boldsymbol{b}(\boldsymbol{l}^{\mathrm{T}}\hat{\boldsymbol{\zeta}}_k + n_k - r_k) + \boldsymbol{S}^{\mathrm{T}}\boldsymbol{w}_k + \mathbf{e}v_{k+1} \\ &= \boldsymbol{A}_{\mathrm{CL}}\hat{\boldsymbol{\zeta}}_k + \boldsymbol{A}\tilde{\boldsymbol{\zeta}}_k + \boldsymbol{b}(n_k - r_k) + \boldsymbol{S}^{\mathrm{T}}\boldsymbol{w}_k + \mathbf{e}v_{k+1},\end{aligned} \tag{34}$$

with $\boldsymbol{A}_{\mathrm{CL}} = \boldsymbol{A} + \boldsymbol{b}\boldsymbol{l}^{\mathrm{T}}$. Noting that the estimate $\hat{\boldsymbol{\zeta}}_k$ and the estimation error $\tilde{\boldsymbol{\zeta}}_k$ are orthogonal, we get

$$\boldsymbol{C}_{\hat{\zeta}} = \boldsymbol{A}_{\mathrm{CL}}\boldsymbol{C}_{\hat{\zeta}}\boldsymbol{A}^{\mathrm{T}}_{\mathrm{CL}} + \boldsymbol{C}^{\mathrm{P}}_{\tilde{\zeta}} - \boldsymbol{C}_{\tilde{\zeta}} + \boldsymbol{b}\boldsymbol{b}^{\mathrm{T}}\frac{c_n^2}{c_n + c_q}. \tag{35}$$

Finally, noting that $t_k = \boldsymbol{l}^{\mathrm{T}}_{\mathrm{OL}}\hat{\boldsymbol{\zeta}}_k$ (cf. Eq. 30), the variance of the transmit signal is $c_t = \boldsymbol{l}^{\mathrm{T}}_{\mathrm{OL}}\boldsymbol{C}_{\hat{\zeta}}\boldsymbol{l}_{\mathrm{OL}}$.

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