# Diversity and Correlation in Rayleigh Fading MIMO Channels

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***Abstract*— The performance of MIMO systems and the usefulness of dedicated signal processing and channel coding techniques is strongly affected by presence of fading correlation. For instance, space-time block coding is favored by low fading correlation, while beam-forming techniques are advantageous at high amounts of correlation. In this paper we propose an easy to compute quantitative measure of fading correlation and a measure of diversity present in a MIMO channel. It turns out that channels with the same amount of diversity or correlation behave essentially the same with respect to channel capacity and throughput. On the other hand, channels which differ considerably in diversity and correlation measure, behave fairly different with respect to channel capacity or throughput. In this way it becomes possible to perform a classification of MIMO channels.**

## I. Introduction

It is known, that fading correlation has significant impact on the performance of MIMO systems. Depending on how much the transmitter is aware of the fading correlation, the effect can be capacity decreasing [2], [3] or even capacity increasing [9]. Fading correlation is directly connected to the diversity gain of a MIMO system. It is also strongly related to MIMO antenna gain [10] and multiplexing gain. The close relation of fading correlation with these three elementary gains provided by a MIMO system, suggests to search for a quantitative description of fading correlation. Such an attempt has already been made in [8] for the SIMO case, which provides a generalized definition of receive diversity order. The authors of [8] compute the ratio of variance of SNR after maximum ratio combining of all received signals and the variance of SNR of a single received signal. In this paper we define a measure of diversity and a measure of correlation for a MIMO channel, which 1) does not depend on maximum ratio combining, 2) is able to separate receive and transmit diversity, and 3) is also applicable in some non-Gaussian fading cases. Interestingly, it turns out that the result from [8] is a special case of the proposed measure of diversity.

Applications of the two proposed measures include: 1) *Quantification* of the amount of correlation and diversity present in the MIMO channel. This allows for instance to decide which of two MIMO channels has stronger correlation or provides more diversity. This method is compatible to the use of majorization [4] proposed in [6], [7]. 2) *Construction of equivalence classes* of MIMO channels which offer the same amount of diversity or correlation. It turns out, that channels having the same correlation measure perform essentially equivalently with respect to channel capacity or throughput. This result may have impact on simulation aspects, as the simulation results obtained for one channel can essentially be used for the whole equivalence class. 3) *Classification of channel types*. Since different transmit and receive signal processing may be used for different amount of correlation, the classification may decide upon the proper signal processing or its parameters (e.g. number of transmitted data streams, modulation schemes, selection between space-time coding and beamforming).

## II. Channel Model

In the following we will assume a frequency flat fading MIMO channel, with $N$ transmit and $M$ receive antennas, described by its channel matrix $\boldsymbol{H} \in \mathcal{C}^{M\times N}$ which is composed of complex, circularly symmetric random variables, which exhibit certain correlations. Let us stack all columns of the channel matrix $\boldsymbol{H}$ into one $K = MN$ dimensional channel vector $\boldsymbol{h} = \mathbf{vec}[\boldsymbol{H}] \in \mathcal{C}^{K\times 1}$, where we have used $\mathbf{vec}[.]$ as the column stacking operation. We can describe the correlations between the random entries of $\boldsymbol{H}$ by the correlation matrix $\boldsymbol{R} = \mathrm{E}[\boldsymbol{h}\boldsymbol{h}^{\mathrm{H}}]$. The channel vector $\boldsymbol{h}$ can then be written as

$$\boldsymbol{h} = \boldsymbol{R}^{\frac{1}{2}}\boldsymbol{g}. \tag{1}$$

where $\boldsymbol{g} \in \mathcal{C}^{K\times 1}$ is a random vector with idependent and identically distributed (i.i.d.) random entries which have zero mean and unity variance. We make the following

*Definition 1: A fading distribution is called* **proper** *if the random entries $g_k = \zeta_k + \mathrm{j}\eta_k$ of the vector $\boldsymbol{g}$ in (1) have the following properties: the $\zeta_k$ and $\eta_k$ are i.i.d. real random variables and:*

$$\begin{aligned} \mathrm{E}\left[g_k\right] &= 0 \\ \mathrm{E}\left[|g_k|^2\right] &= 1 \\ \mathrm{E}\left[|g_k|^4\right] &= 2. \end{aligned} \tag{2}$$

Note that a zero-mean complex circularly symmetric complex Gaussian distribution of the $g_k$ (Rayleigh fading) leads to a proper fading distribution by this definition. Even though other proper fading distributions exist, we will concentrate on Rayleigh fading in the following.

## III. Diversity Measure

*Definition 2: Given a MIMO channel matrix $\boldsymbol{H}$ with correlation matrix $\boldsymbol{R} = \mathrm{E}\left[\mathbf{vec}[\boldsymbol{H}]\mathbf{vec}[\boldsymbol{H}]^{\mathrm{H}}\right]$. The* **diversity measure** *$\Psi(\boldsymbol{R})$ is defined as:*

$$\Psi(\boldsymbol{R}) = \left(\frac{\mathrm{tr}\,\boldsymbol{R}}{||\boldsymbol{R}||_{\mathrm{F}}}\right)^2. \tag{3}$$

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Here the symbol $\mathrm{tr}$ is used for the trace operator, while $||.||_{\mathrm{F}}$ denotes the Frobenius norm.

*Theorem 1: If the fading distribution of the MIMO channel matrix is* **proper** *in sense of Definition 1, the diversity measure* $\Psi(\boldsymbol{R})$ *has the following property:*

$$\Psi(\boldsymbol{R}) = \frac{(\mathrm{E}[\gamma])^2}{\mathrm{var}[\gamma]}, \quad \textit{where} \tag{4}$$

$$\gamma := ||\boldsymbol{h}||_2^2 = ||\boldsymbol{H}||_{\mathrm{F}}^2 . \tag{5}$$

The proof can be found in [11]. Since $\gamma$ is the sum of the squared magnitudes of channel coefficients, it represents the (random) channel energy. The diversity measure $\Psi(\boldsymbol{R})$ therefore quantifies the *relative fluctuation* in channel energy. The diversity measure has several further properties, including

- $1 \le \Psi(\boldsymbol{R}) \le \mathrm{rank}(\boldsymbol{R}) \le K$, where $\boldsymbol{R} \in \mathcal{C}^{K\times K}$.
- Let the first $L$ eigenvalues of $\boldsymbol{R}$ be positive and identical (flat eigenvalue profile) and the remaining eigenvalues vanish, i.e. $\lambda_1 = \lambda_2 = \cdots = \lambda_L > \lambda_{L+1} = \cdots = 0$. Then the diversity measure becomes $\Psi = L$, which is also the rank of $\boldsymbol{R}$. In case the eigenvalue profile is not flat, then $\Psi(\boldsymbol{R}) < L$ might be interpreted as a "soft-rank" of $\boldsymbol{R}$ (number of dominant eigenvalues).
- For an exponential eigenvalue profile given by $\lambda_k = \lambda \cdot a^{k-1}$ for $k = 1, 2, \ldots, K$, with $0 \le a \le 1$, it can be shown that for $K \to \infty$ the diversity measure becomes $\Psi = (1+a)/(1-a)$.
- The diversity measure $\Psi(\boldsymbol{R})$ can also be written in terms of the eigenvalues $\lambda_i$ of $\boldsymbol{R}$:

$$\Psi(\boldsymbol{R}) = \frac{\left(\sum_i \lambda_i\right)^2}{\sum_i \lambda_i^2}. \tag{6}$$

In case $\boldsymbol{R}$ can be decomposed into the tensor product of the *transmit correlation* matrix $\boldsymbol{R}_{\mathrm{Tx}} = \mathrm{E}[\boldsymbol{H}^{\mathrm{H}}\boldsymbol{H}] \in \mathcal{C}^{N\times N}$ and the *receive correlation* matrix $\boldsymbol{R}_{\mathrm{Rx}} = \mathrm{E}[\boldsymbol{H}\boldsymbol{H}^{\mathrm{H}}] \in \mathcal{C}^{M\times M}$

$$\boldsymbol{R} = \frac{1}{\mathrm{tr}\ \boldsymbol{R}_{\mathrm{Tx}}} \boldsymbol{R}_{\mathrm{Tx}}^{\mathrm{T}} \otimes \boldsymbol{R}_{\mathrm{Rx}}, \tag{7}$$

it can be shown [11] that $\Psi(\boldsymbol{R})$ decomposes into the product

$$\Psi(\boldsymbol{R}) = \Psi(\boldsymbol{R}_{\mathrm{Tx}}) \cdot \Psi(\boldsymbol{R}_{\mathrm{Rx}}), \tag{8}$$

of a *transmit* diversity measure $\Psi(\boldsymbol{R}_{\mathrm{Tx}})$ and a *receive* diversity measure $\Psi(\boldsymbol{R}_{\mathrm{Rx}})$. The decomposition of (7) holds if a random realization of the channel matrix can be written as:

$$\boldsymbol{H} = \frac{1}{\sqrt{\mathrm{tr}\boldsymbol{R}_{\mathrm{Tx}}}} \boldsymbol{R}_{\mathrm{Rx}}^{1/2} \boldsymbol{G} \boldsymbol{R}_{\mathrm{Tx}}^{1/2}, \tag{9}$$

where $\boldsymbol{G} \in \mathcal{C}^{M\times N}$ is a complex random matrix with all elements being i.i.d. random variables with proper distribution in the sense of definition 1. This describes a situation where the random fading processes at the receiver are uncorrelated to those at the transmitter.

## IV. Correlation Measure

The measure for diversity in Rayleigh fading MIMO systems, that has been defined in Chapter III is based on correlation matrices, and therefore directly related to the correlation properties of a MIMO channel. In this section, based on the defined diversity measure, we provide a definition of correlation measure.

*Definition 3:* The **Correlation Measure** $\Phi(\boldsymbol{R})$ of a $L \times L$ correlation matrix $\boldsymbol{R}$ is given by

$$\Phi(\boldsymbol{R}) = \sqrt{\frac{1 - L/\Psi(\boldsymbol{R})}{1 - L}} . \tag{10}$$

This generic definition of the correlation measure $\Phi(\boldsymbol{R})$ can be applied to measure receive, transmit or the total correlation depending on the choice of $\boldsymbol{R}$. The motivation stems from the following observation. Let us have a look at a specific correlation matrix $\boldsymbol{R} \in \mathcal{C}^{L\times L}$:

$$\boldsymbol{R} = \begin{bmatrix} 1 & \rho & \rho & \cdots & \rho \\ \rho^* & 1 & \rho & \cdots & \rho \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \rho^* & \rho^* & \rho^* & \cdots & 1 \end{bmatrix} .$$

Since in this case, the correlation properties are essentially captured by the single parameter $\rho$, we want our proposed correlation measure to yield the same result. Since the diversity measure in this situation is

$$\Psi(\boldsymbol{R}) = \frac{L}{1 + (L-1)\cdot|\rho|^2} ,$$

by substituting into (10), we obtain $\Phi(\boldsymbol{R}) = |\rho|$ , as desired.

There are two special cases, we want to bring to the reader's attention. They are characterized by their eigenvalue profile.

1) **Rank one situation**: the correlation matrix has only one non-zero eigenvalue. This represents the strongest possible correlation. The correlation measure from (10) yields

$$\Phi = 1 ,$$

2) **Uncorrelated and equal power situation**: all eigenvalues of the $L \times L$ correlation matrix are identical and positive. The correlation measure this time yields

$$\Phi = 0.$$

Note, that we always have $0 \le \Phi \le 1$. Therefore, by virtue of Definition 3, we have a way to quantify the amount of correlation, from $\Phi = 0$ representing the uncorrelated case, up to $\Phi = 1$ representing maximum correlation. The larger the correlation measure, the stronger is the correlation between channel coefficients of the MIMO channel and the less diversity is available. The correlation measure has some interesting properties:

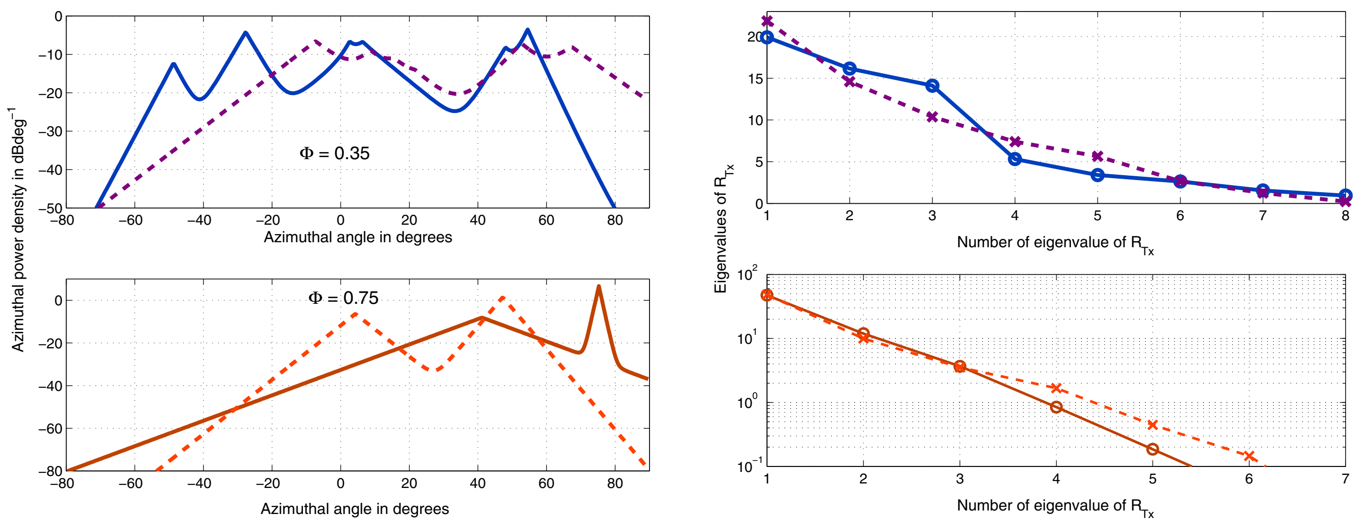


Fig. 1. *Left*: Four different azimuthal transmit power spectra with two different correlation measures. *Right*: Eigenvalue profile of corresponding transmit correlation matrix. A uniform linear array (half wave-length spacing) with $N = 8$ antennas is assumed

- For a correlation matrix

$$\boldsymbol{R} = \begin{bmatrix} 1 & \rho_{1,2} & \rho_{1,3} & \cdots & \rho_{1,K} \\ \rho_{1,2}^* & 1 & \rho_{2,3} & \cdots & \rho_{2,K} \\ \vdots & \ddots & \ddots & \ddots & \ddots \\ \rho_{1,K}^* & \rho_{1,K-1}^*, & \rho_{1,K-2}^* & \cdots & 1 \end{bmatrix}$$

  the correlation measure computes to

$$\Phi = \sqrt{\frac{1}{K(K-1)} \sum_{n=1}^{K} \sum_{\substack{m=1 \\ m \neq n}}^{K} |\rho_{n,m}|^2}.$$

  Hence, the correlation measure is the root mean square of the magnitudes of all correlation coefficients.

- In the general case, where the main diagonal of $\boldsymbol{R}$ contains different values, the situation is more complicated. The correlation measure provides a way to asses such situations. Let us look at the following example:

$$\boldsymbol{R} = \begin{bmatrix} 1 & \rho\sqrt{\eta} \\ \rho^*\sqrt{\eta} & \eta \end{bmatrix},$$

  where $\eta > 0$ and $\rho$ with $|\rho| \leq 1$ is the correlation coefficient. The correlation measure now computes to

$$\Phi(\boldsymbol{R}) = \sqrt{1 + \frac{4\eta}{(1+\eta)^2} \cdot \left(|\rho|^2 - 1\right)}.$$

  We have $\Phi(\boldsymbol{R}) \geq |\rho|$, where equality holds only when $\eta = 1$ or when $|\rho| = 1$. This behaviour makes sense, since having one random variable dominate the other in variance (either $\eta \gg 1$ or $0 < \eta \ll 1$) will intuitively increase average correlation. In the extreme cases of $\eta = 0$ or $\eta \to \infty$ the correlation measure reports the value of $\Phi = 1$, which makes perfect sense, as the correlation matrix then has a rank of one and the random variables become coherent.

- In the case of independent transmit and receive correlation the total correlation measure can be computed directly from the receive and transmit correlation measures:

$$\Phi^2 = \frac{1 - \left(1 - \Phi_{\mathrm{Rx}}^2\,(1-M)\right)\left(1 - \Phi_{\mathrm{Tx}}^2\,(1-N)\right)}{1 - MN}, \qquad (11)$$

  where $M$ is the number of receive and $N$ the number of transmit antennas, while $\Phi_{\mathrm{Rx}}$ and $\Phi_{\mathrm{Tx}}$ are the correlation measures of the receive and the transmit correlation matrices, respectively.

- For large numbers of both receive and transmit antennas there is the asymptotic property $\lim_{N,M\to\infty} \Psi \cdot \Phi^2 = 1$.

## V. Applications

There are several applications of the diversity and correlation measures defined in Chapters III and IV, respectively. They include the following:

- *Establishment of equivalence classes*:
  By grouping together different correlation matrices which have the same diversity or correlation measure, a so called equivalence class is obtained. It turns out that MIMO channels from one such equivalence class offer similar performance in terms of ergodic capacity and throughput. One element out of the equivalence class can then be used as a representative for the whole class. This has impact for physical layer simulation of mobile communication systems, as only a small number of representative channel types have to be simulated.
- *Build an order relation of MIMO channels*:
  The diversity and correlation measures are functions which map a correlation matrix onto a non-negative real number. They can be used to define an order relation according to which correlation matrices can be sorted by

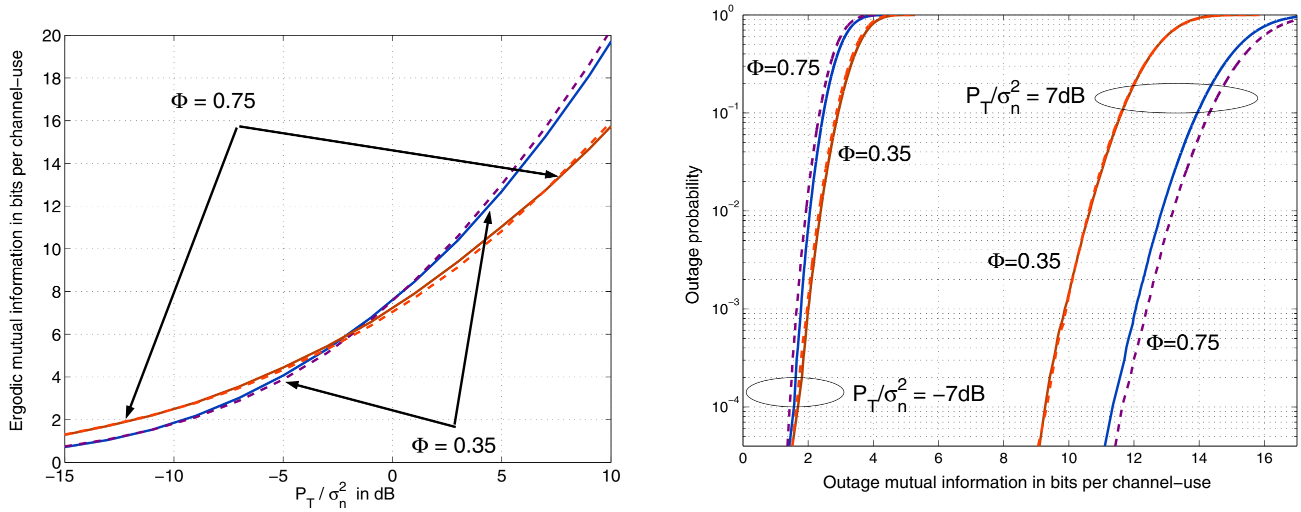


Fig. 2. *Left*: Ergodic mutual information with eigenbeamforming for transmit correlation matrices from azimuthal power spectra from Figure 1. *Right* Outage mutual information for the same transmit correlation matrices.

their amount of correlation. Note that the correlation measure can be directly applied also to compare correlation matrices of different dimensions.

- *Statistical analysis of correlation matrices*:
  Sometimes the correlation matrices are modeled as random variables themselves. The diversity and correlation measures can be used to analyze the statistical properties of diversity and correlation associated with these random correlation matrices. This is especially useful for analysis of data obtained through field measurement.
- *Classification of channel types*:
  The amount of diversity or correlation can be used to classify a MIMO channel. For instance one could define three classes, which collect channels of low, medium and high correlation, respectively. Since different transmit and receive signal processing may be used for different amount of correlation, the classification may decide upon the proper signal processing or its parameters. This may include selection of number of transmitted data streams, associated modulation schemes and distribution of transmit power, as well as selection between transmit processing algorithms which are built on diversity (like space-time block coding) or beam-forming oriented schemes, which profit from higher correlation. Since the diversity and the correlation measures can be computed with both very low and constant complexity, a decision based on this criterion is attractive for real-time applications.

Let us discuss the first application in more detail. The correlation (or diversity ) measure allows the definition of equivalence classes $\mathcal{S}_\Phi$ of correlation matrices, which all have the same correlation measure $\Phi$, that is

$$\mathcal{S}_\Phi \;=\; \{\boldsymbol{R} \mid \Phi(\boldsymbol{R}) = \Phi\}. \tag{12}$$

These equivalence classes have one important feature: the performance of MIMO systems with respect to channel capacity and throughput is essentially equivalent for all correlation matrices out of the equivalence class. In order to demonstrate this property let us first have a look at an example. We assume that the receive and the transmit fading processes are uncorrelated, such that the channel matrix is given by (9), where we let the entries of $\boldsymbol{G}$ become i.i.d. zero-mean, unity-variance Gaussian distributed (Rayleigh fading). A transmit fading correlation matrix, can be specified by

$$\boldsymbol{R}_{\mathrm{Tx}} = \sum_n \boldsymbol{a}(\varphi_n) P(\varphi_n) \boldsymbol{a}^{\mathrm{H}}(\varphi_n), \tag{13}$$

where $\boldsymbol{a}(\varphi_n)$ is the array steering vector of the transmit antenna array corresponding to a azimuthal angle of departure $\varphi_n$, while $P(\varphi_n)$ is proportional to the so-called azimuthal power density at the angle $\varphi_n$. The power spectrum $P(\varphi_n)$ therefore describes the azimuthal scatterer distribution as seen by the transmitter. We look at four different transmit correlation matices which are obtained from the four different azimuthal power densities given on the left hand side of Figure 1. The two power spectra in the upper part lead to correlation matrices with correlation measure equal to $\Phi = 0.35$ (low correlation). In the lower part two different power spectra are given which lead to a considerably higher correlation with a correlation measure of $\Phi = 0.75$. Assuming a uniform linear antenna array with half-wavelength spacing, the right hand side of Figure 1 displays the eigenvalue profiles of the transmit fading correlation matrices corresponding to the power spectra from the left hand side. The transmitter knows the channel on average, i.e. it is aware of the transmit correlation matrices. This enables the application of MIMO eigenbeamforming [9]. For the computation of mutual information we assume that the receiver is also equipped with $N = 8$ antennas, which do not experience fading correlations (rich local scattering around the receiver). Such a channel is called semi-correlated (only transmit-side fading correlation) [10] . The left hand side of Figure 2 shows the ergodic mutual information (assum-

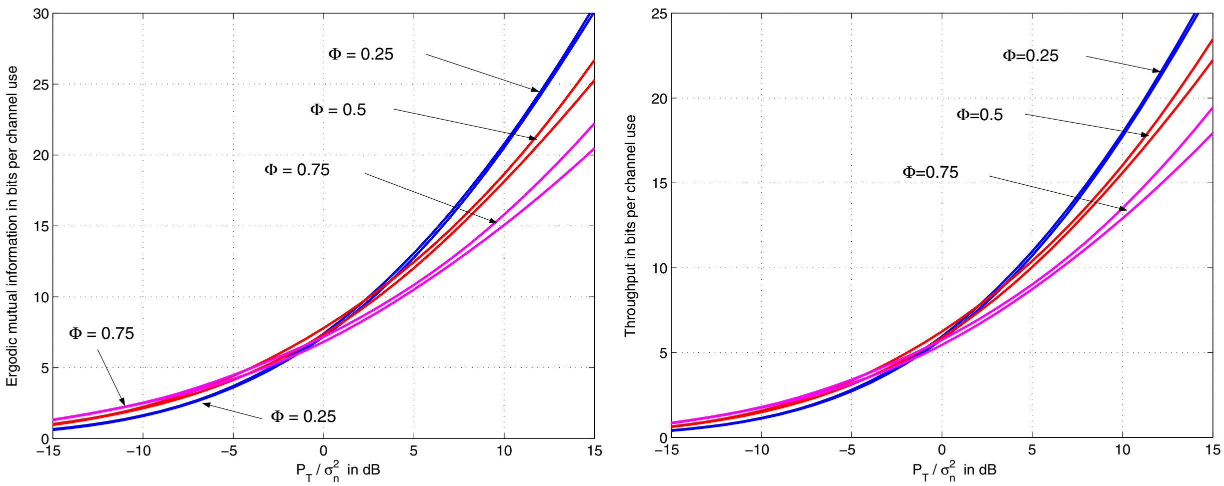


Fig. 3. *Left*: Ergodic mutual information vs. transmit power for transmit correlation matrices from the sets $\mathcal{S}_{0.25}$, $\mathcal{S}_{0.5}$ and $\mathcal{S}_{0.75}$. *Right*: Same for throughput. For each of the sets two lines are shown, which represent the range of $\pm 1$ standard deviation around the average. MIMO-Eigenbeamforming is applied at the transmitter. The number of receive and transmit antennas equals 8 each.

ing eigenbeamforming) which results from the four different transmit correlation matrices. It can be seen that channels with transmit correlation matrices which have the same correlation measure perform essentially the same with respect to ergodic mutual information. Moreover, the performance is fairly different when the transmit correlation matrices have a different correlation measure. The right hand side of Figure 2 shows that this property also holds true for the outage mutual information. The qualitative results from Figure 2 can be generalized to arbitrary transmit correlation matrices. This is shown on the left hand side of Figure 3, which displays the ergodic mutual information (with eigenbeamforming) of MIMO channels with different transmit correlation matrices, which are grouped into three sets $\mathcal{S}_{0.25}$, $\mathcal{S}_{0.5}$ and $\mathcal{S}_{0.75}$. Each set consists of 250 transmit correlation matrices which have the same correlation measure (0.25, 0.5 and 0.75, respectively) but are otherwise chosen randomly. For each of the sets Figure 3 shows two lines, which represent the range of $\pm 1$ standard deviation around the average over the 250 correlation matrices from each set. The right hand side of Figure 3 shows the corresponding results in terms of throughput $T$ which is defined as:

$$T = \max_{R} \begin{cases} R & \text{for } I > R \\ 0 & \text{else} \end{cases} , \qquad (14)$$

where $I$ is the mutual information. As can be seen, the transmit correlation matrices out of the same equivalence class lead to highly similar performance with respect to both ergodic capacity and throughput.

## VI. Conclusion

In this paper a measure for diversity and correlation present in a wireless MIMO communication system is shown. With this definition it is not only possible to quantify the amount of correlation and diversity present in the channel, but also to classify channel types by their amount of diversity or correlation. Thereupon equivalence classes of channels can be defined, which offer the same amount of diversity or correlation. It is demonstrated, that channels with the same correlation measure, i.e. from the same equivalence class perform essentially equivalent with respect to both ergodic capacity and throughput. The correlation measure may also be used for other applications, like ranking the suitability of different signal processing and coding techniques, for instance space-time block-coding or beam-forming in correlated fading. Both the correlation and diversity measure can be computed from the correlation matrix without knowing its eigenvalues.

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